Edward Sang's computation of the logarithms of integers

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Contents

Dedicated to Alex D.D.Craik.

1 Introduction

Edward Sang (1805-1890) was probably the greatest calculator of logarithms of the 19th century [3, 11, 12, 13, 24, 30]. Sang spent 40 years computing tables of logarithms and trigonometric functions, with the assistance from his daughters Flora (1838-1925) and Jane (1834-1878). The result fills about 50 manuscript volumes, plus a number of transfer duplicates.

In 1871, he published a table of 7-place decimal logarithms from 20000 to 200000 [61]. His plans were to publish a 9-place table of decimal logarithms from 100000 to one million based on entire new computations and not on previously published tables [31]. The latter was never published, but, like the 7-place table, it would have been based on a 15-place table of decimal logarithms, which Sang completed from 100000 to 370000. This table spans 27 volumes of 10000 logarithms each, a total of 10800 pages. Sang's manuscripts are kept in part at the National Library of Scotland and in part at Edinburgh University Library.

Sang's purpose was in particular to provide fundamental tables, including for the decimal division of the quadrant. In 1890 [3, p. 189], he wrote that

> In addition to the results being accurate to a degree far beyond what can ever be needed in practical matters, [the collection of computations] contains what no work of the kind has contained before, a complete and clear record of all the steps by which those results were reached. Thus we are enabled at once to verify, or if necessary, to correct the record, so making it a standard for all time.
>
> For these reasons it is proposed that the entire collection be acquired by, and preserved in, some official library, so as to be accessible to all interested in such matters; so that future computers may be enabled to extend the work without the need of recomputing what has been already done; and also so that those extracts which are judged to be expedient may be published.

My purpose here is to analyze how Sang computed the logarithms of numbers and to trace the paths of future research on Sang's tables.

2 The manuscripts

In order to facilitate the use of Sang's volumes, I refer to the volumes in Knott's inventory [3], as K1, K2, K3, etc. K1 corresponds to the signature Acc.10780/16 in the National Library of Scotland, K2 corresponds to Acc.10780/17, and so on, until Acc.10780/21 [29].[1] Volumes K7 to K38 are the 15-place table. There are other tables in the subsequent volumes, but we are not concerned with them here. Thus, we have the following general layout:

- K1: Construction (introduction of 13 pages, construction pages 1-240, summary of the logarithms of primes 48 pages)
- K2: Construction (pages 241 to 536)
- K3: Construction (pages 537 to 770)
- K4: Logarithms of the first 10000 primes, but to 28 places only until 10037, and a few afterwards
- K5: Logarithms of the first 10000 integers to 28 places
- K6: Logarithms of 10000 to 20000 to 28 places or 13-14 places for new primes
- K7 to K38: 15-place logarithms from 100000 to 370000, 10000 logarithms per volume, the range 100000-150000 being computed twice (hence $27 + 5 = 32$ volumes)

I have recorded all of the steps used by Sang and have reconstructed an approximation of the above volumes giving the computation of the logarithms of the primes (volumes K1, K2, K3) [43].

I have also reconstructed Sang's table of the logarithms of the first 10000 primes (K4, up to 104759 which is the 10002nd prime), and of the first (K5) and second (K6) myriad of integers [44, 45, 46].

Finally, I have reconstructed the (exact) 15-place table up to one million [47], and this should make it easier for future researchers to assess the accuracy of Sang's manuscripts.

In the sequel, I am describing how Sang computed the logarithms from 1 to 20000, what are the problems of computing other logarithms from an initial set of logarithms, how the 15-place table of logarithms was computed, and the reduced tables which have been obtained from them.

[1] For more information on signatures and locations, see my guide on Sang's tables and their reconstruction [41].

3 Logarithms from 1 to 20000

The computation of the logarithms from 1 to 20000 was split in two parts. First, Sang computed the logarithms of the primes up to 10000, then the logarithms of the other integers.

3.1 The computation of the logarithms of the primes

Sang computed these logarithms incrementally, following a cumulative path, where each logarithm helps to find the logarithms of new numbers, and sometimes of primes.

This process is very different from the more systematic computations of Sang's predecessors, with the exception of Isaac Wolfram (see [40]).

As Sang stresses in an article published in 1878 [69], it is important that all the steps leading to the calculation of a logarithm be recorded, because this is then what will enable someone who stumbles upon an error to trace the origin of this error. As a comparison, Sang observes that there is no traceability for the 61-place logarithms given by Sharp [76], although these logarithms appear in retrospect exempt of any error.

3.1.1 Computing M

Sang's computations rely on the value of $M = \frac{1}{\ln 10}$ and on the development of $\ln(1+x)$. Sang first computed M to 28 places. This number of places was chosen in order to compensate for losses of accuracy during the various steps of the computation. However, as I will show later on, Sang hardly ever used this accuracy for his final 15-place table.

In any case, Sang proceeded as follows. In order to compute $\ln 10$, he considered the two series

$$\ln 8 = 2\left[1 + \frac{1}{3}\cdot\frac{1}{9} + \frac{1}{5}\cdot\frac{1}{9^2} + \frac{1}{7}\cdot\frac{1}{9^3} + \cdots\right]$$
$$\ln\frac{10}{8} = 2\left[\frac{1}{9} + \frac{1}{3}\cdot\frac{1}{9^3} + \frac{1}{5}\cdot\frac{1}{9^5} + \frac{1}{7}\cdot\frac{1}{9^7} + \cdots\right]$$

obtained from $\ln\left(\frac{1+a}{1-a}\right) = 2a\left[1 + \frac{a^2}{3} + \frac{a^4}{5} + \frac{a^6}{7} + \cdots\right]$ with $a = \frac{1}{3}$ and $\frac{1}{9}$, and by adding these two series, he obtained a series for $\ln 10$ [12, p. 75]:

$$
\begin{aligned}
\ln 10 &= 2 + \frac{2}{9}\left(1+\frac{1}{3}\right) + \frac{2}{5}\cdot\frac{1}{9^2} + \frac{2}{9^3}\left(\frac{1}{3}+\frac{1}{7}\right) + \frac{2}{9}\cdot\frac{1}{9^4} + \frac{2}{9^5}\left(\frac{1}{5}+\frac{1}{11}\right)\\
&\quad + \frac{2}{13}\cdot\frac{1}{9^6} + \frac{2}{9^7}\left(\frac{1}{7}+\frac{1}{15}\right) + \frac{2}{17}\cdot\frac{1}{9^8} + \frac{2}{9^9}\left(\frac{1}{9}+\frac{1}{19}\right) + \cdots\\
&= \sum_{n=0}^{\infty}\left[\frac{2}{4n+1}\cdot\frac{1}{9^{2n}} + \frac{2}{9^{2n+1}}\left(\frac{1}{2n+1}+\frac{1}{4n+3}\right)\right]
\end{aligned}
$$

Sang possibly computed this series on 28 places until $n = 13$, and then computed $M = 1/\ln 10$ which he gave correctly to 27 places:

$$M = 0.43429\ 44819\ 03251\ 82765\ 11289\ 19$$

The correct value to 32 places is actually

$$M = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18916\ 60\ldots$$

More accurate values of M had been published before. For instance in 1794 Wolfram gave 48 places correctly rounded in Vega's *Thesaurus* [78, p. 641], but it was important for Sang to recompute the value of M *ab novo*.

3.1.2 The first logarithms

Let us see how Sang computes the first logarithms. Let

$$P(x) = Mx + Mx^3/3 + Mx^5/5 + \ldots$$

and

$$N(x) = Mx^2/2 + Mx^4/4 + \ldots$$

We then have $\log(1+x) = M\ln(1+x) = P(x) - N(x)$.

1. Sang first takes $x = 1/10$:

   (a) On page K1/6, Sang computes $P(x)$ up to $Mx^{25}/25$ and $N(x)$ up to $Mx^{26}/26$, and then $P(x) - N(x) = M\ln(1+x) = \log(1+x)$ The sums of the positive terms and negative terms are computed separately to 28 places. So, Sang computes

   $$\log(1+1/10) = \log(11/10) = \log(11) - 1$$

   and page 7 has
   $\log(11) = 1.04139\ 26851\ 58225\ 04075\ 01999\ 711$
   which is off by one unit at the last place.

(b) Sang also computes $P(x) + N(x) = 0.0457574905... = -\log(1 - x) = \log(10/9) = 1 - \log(9)$ from which Sang deduces $\log(9) = 0.95424\ 25094...$

(c) and then he obtains $\log(3) = \log(9)/2 = 0.47712\ 12547\ ...$

(d) and $\log(81) = 2\log(9) = 1.90848\ 50188\ ...$

2. Then, taking $x = 1/80$, Sang computes Mx, Mx^2, Mx^3, Mx^4, $\ldots$, Mx^{14}

   (a) (page K1/8)
   from which he computes $P(x)$ and $N(x)$, and subtracts them to obtain $\log(81/80) = 0.00539\ 50318\ ...$ and using the value of $\log(81)$ found previously, he gets $\log(80) = \log(81) - \log(81/80) = 1.90308...$

   (b) Adding $P(x)$ and $N(x)$, Sang obtains $\log(80/79)$, from which $\log(79)$ is derived.

   (c) Since $\log(80) = \log(2^3 \times 10) = 1 + 3\log 2$, Sang can immediately compute $\log 2$ from $\log 80$.

   (d) And since $\log 10 = \log 2 + \log 5$, we have $\log 5 = 1 - \log 2$.

   (e) Then, Sang computes $\log 2400 = \log(2^3 \times 3 \times 100) = 2 + 3\log 2 + \log 3$.

3. (page K1/9) Sang then takes $x = 1/2400$, and derives $\log 2399$ and $\log 2401$. And since $2401 = 7^4$, he obtains also $\log 7$.

At this point, Sang has obtained the values of log(2), log(3), log(5), log(7), log(9), log(10), log(11), log(79), log(80), log(81), log(2399), log(2400), and log(2401), merely using three different values of x.

This process goes on, with adequate choices for x and little by little the logarithms of all primes up to 10000 are found. The next logarithms found are those of 999, 1001, 13, 27, 37, 399, 401, 400, 133, 19, 31499, 31501, etc. I will show in section 3.1.4 how x is chosen to yield the logarithm of a sought prime.

3.1.3 The general procedure

Sang's method to compute the logarithm of a prime number involves finding an equation relating this prime to primes whose logarithms have already been computed and to a number differing by one from a number ending with several 0s [12]. In the first calculations, 2399 and 2401 are such numbers.

In order to compute $\log 8447$, Sang may for instance have considered the equation[2]

$$2 \cdot 3769 \cdot 10^8 + 1 = 3 \cdot 37 \cdot 251 \cdot 3203 \cdot 8447 \tag{1}$$

where the logarithms of 2, 3, 37, 251, 3203, and 3769 were assumed to be already known. It should be noted that the logarithms need not be computed in an increasing order, and if $\log 8447$ had already been known, the previous equation would also make it possible to compute $\log 3203$ from the other known logarithms.

Now, Sang would write

$$\begin{aligned}\log(2 \cdot 3769 \cdot 10^8 + 1) &= \log\left(\frac{2 \cdot 3769 \cdot 10^8 + 1}{2 \cdot 3769 \cdot 10^8}\right) + \log(2 \cdot 3769 \cdot 10^8) \\ &= \log\left(1 + \frac{1}{2 \cdot 3769 \cdot 10^8}\right) + \log 2 + \log 3769 + 8 \log 10\end{aligned}$$

and

$$\begin{aligned}\log 8447 &= \log(2 \cdot 3769 \cdot 10^8 + 1) - \log 3 - \log 37 - \log 251 - \log 3203 \\ &= \log\left(1 + \frac{1}{2 \cdot 3769 \cdot 10^8}\right) + \log 2 + \log 3769 + 8 \log 10 \\ &- \log 3 - \log 37 - \log 251 - \log 3203\end{aligned}$$

$\log\left(1 + \frac{1}{2 \cdot 3769 \cdot 10^8}\right)$ can easily be computed, using the familiar development of $\ln(1+x)$ and a value of $M = 1/\ln 10$. It is in order to ease the computation of $\ln(1 + x)$ that Sang chose $\frac{1}{x}$ to be an integer ending with as many 0s as possible. The divisions are then much easier to perform. Moreover, Sang made use of Burckhardt's factor tables [9] in order to factor the numbers $n \times 10^m \pm 1$ appearing in the above method.

In general, Sang would find several equations for each new prime, so that he could perform independent calculations and avoid errors.

3.1.4 Finding equations for a prime

We have seen above that for the prime 8447, Sang might have resorted to the equation:

$$753800000001 = 89238783 \times 8447 \tag{2}$$

and that this equation could be factored as

$$2 \cdot 3769 \cdot 10^8 + 1 = 3 \cdot 37 \cdot 251 \cdot 3203 \cdot 8447 \tag{3}$$

[2] This equation appears on page K2/255.

Given the prime number 8447, we can notice that we have

$$8447 \times 3 = 25341$$
$$8447 \times 83 = 701101$$
$$8447 \times 783 = 6614001$$
$$8447 \times 8783 = 74190001$$

and so on. The sequence 3, 83, 783, 8783, etc., is constructed incrementally, so as to make an additional 0 appear in the result. The first digit, 3, is found because it is the only integer smaller than 10 which, when multiplied by 7 (the last digit of 8447), ends with 1. The second digit, 8, is found because it is the only integer smaller than 10 which, when multiplied by 7, ends with 6. In that case $83 \times 8447 = 80 \times 8447 + 25341$ and 80×7 will add 60 to the previous product 25341, hence bring this product to end with 01. So, the choice of the digit 8 is based on the complement 6 needed to bring 4 to 0. The third digit, 7, is found because it is the only integer smaller than 10 which, when multiplied by 7, ends with $10 - 1 = 9$. And so on.

One may wonder if it is always possible to let 0s appear one after the other. The answer is yes. First, it is easy to see that the first step (obtaining a final 1) is always possible. Then, observing that primes (greater than 5) can in fact end only with 1, 3, 7, or 9, and multiplying any of these four endings with 0 to 9, we obtain again every terminal digit 0 to 9 (although multiplying by 0 will never be necessary, because that would mean that we have already obtained the sought digit 0):

$1 \times 0 = 0$	$3 \times 0 = 0$	$7 \times 0 = 0$	$9 \times 0 = 0$
$1 \times 1 = 1$	$3 \times 1 = 3$	$7 \times 1 = 7$	$9 \times 1 = 9$
$1 \times 2 = 2$	$3 \times 2 = 6$	$7 \times 2 = 14$	$9 \times 2 = 18$
$1 \times 3 = 3$	$3 \times 3 = 9$	$7 \times 3 = 21$	$9 \times 3 = 27$
$1 \times 4 = 4$	$3 \times 4 = 12$	$7 \times 4 = 28$	$9 \times 4 = 36$
$1 \times 5 = 5$	$3 \times 5 = 15$	$7 \times 5 = 35$	$9 \times 5 = 45$
$1 \times 6 = 6$	$3 \times 6 = 18$	$7 \times 6 = 42$	$9 \times 6 = 54$
$1 \times 7 = 7$	$3 \times 7 = 21$	$7 \times 7 = 49$	$9 \times 7 = 63$
$1 \times 8 = 8$	$3 \times 8 = 24$	$7 \times 8 = 56$	$9 \times 8 = 72$
$1 \times 9 = 9$	$3 \times 9 = 27$	$7 \times 9 = 63$	$9 \times 9 = 81$

We can therefore select a unique factor based on the required complement. The same is true if we want to obtain only 9s, or even any other number.

For instance, a multiple of 8447 could give the first digits of π:

$$\begin{aligned} 8447 \times 5 &= 42235 \\ 8447 \times 45 &= 380115 \\ 8447 \times 945 &= 7982415 \\ 8447 \times 7945 &= 67111415 \\ 8447 \times 67945 &= 5739\underline{31415} \end{aligned}$$

so we have a product ending with 31415, but of course there are additional digits at the beginning.

Going back to the sequence

$$\begin{aligned} 8447 \times 3 &= 25341 \\ 8447 \times 83 &= 701101 \\ 8447 \times 783 &= 6614001 \\ 8447 \times 8783 &= 74190001 \end{aligned}$$

the numbers x on the right side will only be used to compute $\ln(1 + \frac{1}{x-1})$. For instance, with the fourth equation, we would compute $\ln\left(1 + \frac{1}{74190000}\right)$. The leading digits, here 7419, may or may not be factored, depending on the case.

However, the chosen multiplier, among 3, 83, 783, 8783, etc., needs to be factored and involve only primes whose logarithms are already known.

In general, Sang would try to go as far as possible in order to have a large number ending with $0\ldots01$, or $9\ldots9$, with a prefix involving factors whose logarithms are known, *and* a multiplier with prime factors whose logarithms are also already known. If this cannot be achieved, Sang would try to factor the complements of the multipliers to 10, 100, 1000, 10000, etc.:

$$\begin{aligned} 8447 \times 7 &= 59129 \\ 8447 \times 17 &= 143599 \\ 8447 \times 217 &= 1832999 \\ 8447 \times 1217 &= 10279999 \end{aligned}$$

and so on.

We have an immediate correspondence between the two sequences, and the sequence of digits of the multipliers need only be obtained once, not twice. The products can also be derived in a straightforward way:

$$\begin{aligned}
8447 \times 7 &= 8447 \times 10 - 25341 \\
8447 \times 17 &= 8447 \times 100 - 701101 \\
8447 \times 217 &= 8447 \times 1000 - 6614001 \\
8447 \times 1217 &= 8447 \times 10000 - 74190001
\end{aligned}$$

If a prime could not be used, for instance because it would require a yet unknown logarithm, Sang would try another prime, until all primes up to 10000 were computed. In addition, the method above obviously produces several equations for the same prime.

For instance, for 3371, Sang uses at one time or another five different equations:

$$\begin{aligned}
3371 \times 10531 &= 35500001 \\
3371 \times 510531 &= 1721000001 \\
3371 \times 9469 &= 31919999 \\
3371 \times 89469 &= 301599999 \\
3371 \times 489469 &= 1649999999
\end{aligned}$$

These equations are of course clearly related.

All these steps are given by Sang in the volumes K1, K2 and K3. I have recorded all the equations used during the construction of the primes and produced a summary of the steps in the reconstruction of these volumes.

3.2 The logarithms up to 20000

The logarithms up to 20000 are given in volumes K4, K5, and K6. These volumes have references to pages in volumes K1, K2, K3, for the construction of the primes up to 10000. For instance, for prime 2309, there is a reference to page 283 and on page 283 (K2) we find the use of the equation

$$24 \times 10^8 - 1 = 239 \times 4349 \times 2309.$$

There may, however, be cases where the logarithm obtained is not obtained straightforwardly from the equation given. The details will have to be completed by a future examination and a careful recording of the entire construction. Currently, partial recordings were made, but the references in K4, K5, K6 were entirely copied, leading to possible discrepancies that I have not tried to fix at the moment.

3.2.1 The logarithms of the first 10000 primes

The logarithms of the first 10000 primes were meant to be given in volume K4. They are given to 28 places from 2 to 10037. Afterwards, they are only given to 28 places for a few primes. The other values are given to 15 places, or not given at all.

The value of each 28-place logarithm is given with references to its computation, that is with the corresponding page numbers in volumes K1, K2, or K3. Usually only one page number is given, and only seldom two or three. These page numbers were copied by hand from the original volume and should be compared to my approximate reconstructions of volumes 1, 2 and 3.

The 15-place logarithms were taken from the table of 15-place logarithms described below.

3.2.2 The logarithms of the integers 1-10000

The logarithms of the integers from 1 to 10000 are given in volume K5 to 28 places. The logarithms of the composite numbers were obtained from the logarithms of the primes by addition. No interpolation was needed.

3.2.3 The logarithms of the integers 10000-20000

The logarithms of the composite numbers from 10000 to 20000 are given in volume K6 and were obtained from those of volume K5 by addition. For instance, $\log 15264 = \log 2 + \log 7632$. The logarithms of a few primes which had been computed to 28 places are also given. The logarithms of the other primes are given to 13 or 14 places and were taken from the 15-place table described below.

For instance, Sang's table contains the 13 or 14-place values for the logarithms of the first primes after 10037:

number	logarithm
10039	4.00169 04542 3215
10061	4.00264 11490 000
10067	4.00290 00686 1138
10069	4.00298 63408 5678
...	...

These logarithms would presumably have been interpolated to a higher accuracy in the future.

For more details on these three volumes K4, K5 and K6, see my reconstructions [44, 45, 46] and their introductions.

4 The problem of computing other logarithms

So far, I have described how Sang computed the logarithms of the primes up to 10000, as well as a few others beyond (volumes K1, K2, K3, K4), the logarithms of all integers up to 10000 (volume K5), and finally the logarithms of the integers from 10000 to 20000 (volume K6), sometimes only to 13 or 14 places. The question is now how to go beyond.

4.1 Additions and interpolations

Sang had the purpose of computing the logarithms from 100000 up to one million, in view of forming an accurate 9-place table of logarithms, which he actually never completed. Now, there are many integers in the range from 100000 to one million whose logarithms can be computed by mere additions of logarithms of primes below 10000. There are in fact exactly 547442 such integers. For these logarithms, the situation is pretty simple, the only issue being the loss of accuracy during the additions. We therefore need to examine what accuracy can be expected in these cases.

But there are also about 350000 integers which are either prime or whose largest factor is greater than 10000 and whose logarithms have therefore not been computed, with a few exceptions.[3]

In these cases, if no new computation of a logarithm is to be performed, the gaps can be filled by interpolation. That is, we are going to compute approximations of logarithms from nearby known logarithms. This in turn can be done in a variety of ways.

How did Sang proceed? In fact, we do not know all the details of his calculations, and this is compounded by the fact that Sang worked at the same time on the computation (mostly sequentially) of the logarithms of primes and on the computation of the 15-place table. For instance, we know that in 1872 he had computed the logarithms of primes up to 2600, but also

[3] The only primes greater than 10000 whose logarithms have been computed to 28 places are 10007, 10009, 10037, 10091, 10333, 10433, 10559, 10601, 11383, 11579, 11807, 13001, 14243, 14723, 15601, 16087, 16091, 16111, 17027, 17431, 18797, 19403, 19697, 19801, 20071, 21001, 22787, 22807, 22877, 24001, 26407, 26699, 26717, 26801, 32069, 32999, 35999, 37619, 41579, 49999, 50111, 53617, 57143, 59999, 67619, 71999, 77647, 79999, 92857, 93001, 96001, 98999, 103333, 106087, 108421, 109001, 132001, 132857, 137143, 150001, 182353, 190909, 222857, 239999, 252001, 270001, 280001, 353333, 450001, 599999, 799999, 900001, 909091, 1770001, 2100001, 2400001, 2999999, and perhaps a few others not recorded after page 218 of volume K1 (see the reconstruction of volumes K1, K2, and K3). There might also be 14143, as volume K6 gives this logarithm to 28 places with a reference fo page 111 of K1, but on that page I have not recorded the computation of $\log 14143$. This should therefore be checked.

the 15-place table up to 260000 (which may be a coincidence). It is likely, however, that Sang had gaps in the latter, since a number of logarithms involve primes greater than 2600. One would even expect that whenever a new logarithm was computed to 28 places, that the logarithms of all its multiples were added to the 15-place table. For instance, when $\log 2609$ was computed, the logarithms of its multiples in the range 100000-370000 could also be computed, namely $\log(39\cdot 2609)$, $\log(40\cdot 2609)$, $\ldots$, $\log(141\cdot 2609)$, and the gaps could be little by little filled. But in fact, the gaps were likely not as extensive as one might expect, because Sang appears to have computed a number of logarithms by interpolation, even though eventually he could have computed them using known logarithms of primes. He presumably did so in order to save time.

I do not know if Sang went beyond 370000 for the 15-place table, but it is possible that parts of the table were constructed beyond 370000, and never made it into archives. We know however that Sang had already reached 370000 in 1878, as this is asserted in the notice introducing his table of logarithmic sines and tangents [52]. We also know that by April 1874 he had reached 320000 [65].[4] It is likely that Sang didn't make any progress on the table of logarithms of numbers between 1875 and his death, and instead concentrated on other tables. Indeed, Sang worked from around 1875 first on the canon of sines, then on the canon of logarithmic sines and tangents, which was completed in 1888.

In 1871, Sang had also published a 7-place table of logarithms from 20000 to 200000 [61]. He did not explain how he obtained the table, but by 1871 he certainly had computed many of the logarithms from 100000 to 200000 to 15 places. In principle, Sang could not merely derive the 7-place table from the 15-place table then under construction, because there were gaps, but it is possible that the 7-place table was an incentive for Sang to fill the gaps in the range 100000-200000 by interpolation, even for those values which eventually could be computed from logarithms of primes. Perhaps Sang considered that the interpolated values were first approximations, and that eventually these values could be checked again using the logarithms of primes, once their computation was complete.

We have however some details on how Sang filled the gaps of the logarithms of integers whose logarithms of factors were not known, or not yet known.

According to Sang [3, p. 191], the logarithms of the numbers from 100000

[4]In his survey of Sang's work [12, p. 71], Craik dated the 15-place table to the years 1869-1873, but this range should be slightly extended on both sides. And Sang's paper on Vlacq's errors, although appended to a 1874 version of his 1872 specimen of 9-place tables, is really from 1874.

to 150000 were obtained from the logarithms of the primes computed earlier and the intermediate values were obtained by "interpolation of second differences." The logarithms from 150000 to 200000 were obtained by interpolating two terms between the even numbers of the logarithms from 100000 to (presumably) 133333 and adding log(1.5). And finally, the logarithms from 200000 to 370000 were obtained by interpolating one term between the values obtained from 100000 to (presumably) 185000 and adding $\log 2$. Starting with the range 100000 to 150000, this procedure actually makes it possible to compute the logarithms up to 450000 ($150000 \times 1.5 \times 2$).

It is not clear if Sang did really proceed this way, because there would then be serious losses of accuracy when interpolating between values which are themselves interpolations.

We must therefore examine the accuracy that Sang could have obtained by the above method, given that beyond 150000 every computation would use only 15 places. And we must see how Sang would have gone beyond 370000.

On the other hand, we should examine how the 28-place logarithms could have been best used with as little losses of accuracy as possible.

4.2 The loss of accuracy

Sang's computations involved computing the logarithms of primes, and deducing other logarithms either by adding the logarithms of primes, or by interpolation. Either process leads to some loss, but it is important to have a better idea of how much accuracy is lost.

Adding logarithms does in fact not cause much loss. If we assume that the logarithms of all primes below 10000 were computed correctly to 28 places, and if we compute with these values the logarithms of all integers between 100000 and one million involving only factors smaller than 10000, we obtain for instance

```
100081: 41*2441
        1.6127838567197354945094118500 [=log(41)]
        3.3875677794171886085768487834 [=log(2441)]
        5.0003516361369241030862606334 (sum)
        5.0003516361369241030862606334 (exact)
        0.0000000000000000000000000000 (difference)
100082: 2*163*307
        0.3010299956639811952137388947 [=log(2)]
        2.2121876044039578076400914359 [=log(163)]
        2.4871383754771864847546084365 [=log(307)]
```

```
5.0003559755451254876084387671 (sum)
5.0003559755451254876084387672 (exact)
0.0000000000000000000000000001 (difference)
```

Here, $\log 100081$ was computed exactly, whereas $\log 100082$ is off by one unit of the 28th place.

Assuming that the base logarithms are exact, the number of correct logarithm values, and the number of values off by 1, 2, 3, 4, and 5 units of the last place are

last-place units	number
0	264663
1	250937
2	29639
3	2241
4	155
5	7

This covers all composite numbers with factors no greater than 100000, and it appears that the errors do not go beyond the last place and never reach more than 5 units. One such case is the following

```
177147: 3^11
        0.4771212547196624372950279033 [=log(3)]
        ...
        0.4771212547196624372950279033 [=log(3)]
        5.2483338019162868102453069363 (sum)
        5.2483338019162868102453069358 (exact)
        0.0000000000000000000000000005 (difference)
```

Here the error on the result is large, because on one hand we have $\log 3 = 0.47712125471966243729502790325\underline{55}\ldots$ and the error on the rounded value of $\log 3$ is almost equal to its maximal value, namely half a unit of the 28th place, and on the other hand this factor appears a large number of times. $531441 = 3^{12}$ also leads to 5 units off, and larger exponents only occur with powers of 2, but the dropped digits of $\log 2$ only amount to about a quarter of a unit, and hence don't lead to final errors of more than 5 units of the 28th place.

However, if one of the composing logarithms were off by one unit, these errors might accumulate more and we might have a result off by one or two units of the 27th place.

4.3 A summary of interpolation

Before examining in more detail what kind of interpolation Sang may have performed, I need to give an overview of the process of interpolation.[5]

In general, if Δ^1, Δ^2, Δ^3, etc., are the first, second, third, etc., differences of an equidistant series of terms A, B, C, etc., an interpolation between the terms A and B is given by

$$z_x = A + x\Delta^1 + x \cdot \frac{x-1}{2}\Delta^2 + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3}\Delta^3 + \cdots \tag{4}$$

with $z_0 = A$ and $z_1 = B$.

This is Newton's forward difference formula, the differences Δ^n being forward differences.[6] Such an interpolation can go beyond the second term B and in fact we might then view this process as an extrapolation.

If the second differences Δ^2 are constant, the previous expression reduces to

$$z_x = A + x\Delta^1 + x \cdot \frac{x-1}{2}\Delta^2 \tag{5}$$

The differences Δ^1, Δ^2, etc., may be obtained from the actual values of the series, but they may also be computed exactly. In the case of the logarithms of numbers, the first four differences can be computed as follows [36]:

$$\Delta \ \log n = \quad M\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \frac{1}{6n^6} + \frac{1}{7n^7} - \cdots\right] \tag{6}$$

$$\Delta^2 \log n = -M\left[\frac{1}{n^2} - \frac{2}{n^3} + \frac{7}{2n^4} - \frac{6}{n^5} + \frac{31}{3n^6} - \frac{18}{n^7} + \cdots\right] \tag{7}$$

$$\Delta^3 \log n = \quad M\left[\frac{2}{n^3} - \frac{9}{n^4} + \frac{30}{n^5} - \frac{90}{n^6} + \frac{258}{n^7} - \cdots\right] \tag{8}$$

$$\Delta^4 \log n = -M\left[\frac{6}{n^4} - \frac{48}{n^5} + \frac{260}{n^6} - \frac{1200}{n^7} + \cdots\right] \tag{9}$$

I also gave the fourth difference, in order to have an idea of what terms are neglected when the fourth difference is not used.

[5] For a comprehensive analysis of interpolation and estimations of their accuracy, see the descriptions of the tables of Briggs [6] and Prony [36].

[6] Other interpolation formulæ use backward or central differences.

These are the differences for a step of 1 between the terms:

$$\begin{aligned}
\Delta\ \log n &= \log(n+1) - \log(n)\\
\Delta^2 \log n &= \Delta \log(n+1) - \Delta \log n\\
\Delta^3 \log n &= \Delta^2 \log(n+1) - \Delta^2 \log n\\
&\cdots\cdots
\end{aligned}$$

However, in his interpolations, Sang also uses the steps 1/2 and 2/3:

- When the step is 1/2, it is easy to see that

$$\log(n+\frac{1}{2}) - \log(n) = \log(2n+1) - \log(2n) = \Delta \log(2n),$$

  and similarly the new second, third, etc., differences at n are $\Delta^2 \log(2n)$, $\Delta^3 \log(2n)$, etc.

- When the step is 2/3, the first, second, third, etc., differences are $\Delta \log(\frac{3}{2}n)$, $\Delta^2 \log(\frac{3}{2}n)$, $\Delta^3 \log(\frac{3}{2}n)$, etc.

- In general, when the step is s, the differences at n are the (1-)differences at $\Delta \log(n/s)$, $\Delta^2 \log(n/s)$, $\Delta^3 \log(n/s)$, etc.

Sang may have used any of these techniques in his computations.

5 The 15-place table of logarithms

5.1 Introduction

Sang computed the logarithms to 15 places for the integers 100000 to 370000 (volumes K7 to K38) but the computation was intended to reach one million. This computation was done in parallel with the computation of the logarithms of primes to 28 places, but it was apparently started relatively late. According to Craik who examined the original volumes of tables,[7] the construction of the 15-place table took place between 1869 and 1873 [12, p. 71]. However, volumes K7 to K11, which are the first set for the range 100000-150000 carry at least the dates 1866 and 1867, and were likely completed during the years 1865 to 1868. The first date that Craik mentions seems to be the date found in volume K12, which is the first volume for the second computation of the range 100000-150000. On the other hand, Sang

[7] I have mostly examined the transfer duplicates.

had not yet reached 370000 in 1874 [65]. It seems therefore safe to assert that the construction of the 15-place table took place between 1865 and 1875.

From the samples I have taken, the 15-place table seems quite accurate, with errors of probably no more than 3 units in the last place. But in his account drawn in 1890 [3], Sang writes that the errors do not exceed 5 units in the 15th place, and therefore there may be such cases which are not found in my samples.

In the following sections, I study the formation of Sang's 15-place table, on three different intervals: the initial range 100000-150000, the range 150000-200000, and the range 200000-370000. These ranges were implicitely given by Sang in the account drawn up in July 1890 [74] and reprinted in 1908 [3].

5.2 Range 100000-150000 (volumes K7-K16)

In this range, the logarithms were either computed from the logarithms of primes smaller than 10000, or through interpolations. There are however two different copies of the range 100000-150000, corresponding to volumes K7-K11 and K12-K16 [74]. The latter set is meant to be a copy of the former, but in order to avoid copying errors, Sang only copied initial values and the last two figures of the second differences, presumably on each page, and the logarithms were recomputed by integration. Then, the values of the two copies were compared and should be identical in case there were no copying errors and no integration (adding) errors.

One would expect that the logarithms of primes have all been used where necessary in this range, but this is actually not the case. Interpolations were used more than strictly necessary, and many details remain sketchy. I am first going to consider a number of questions on the values given in this range, before giving some conclusions.

5.2.1 Are all logarithms derivable from primes correct?

One would expect that all those logarithms that could be obtained from primes under 10000 are correctly given in Sang's 15-place table. Surprisingly, this is not the case. For instance, $\log 119975 = \log(5^2 \cdot 4799)$ is given as 5.079090758604458, when it should in fact be 5.079090758604459. This is not an anecdotical error, because the logarithms of 5 and 4799 were eventually computed to 28 places. Therefore, the value of $\log 119975$ should be correct to 27 or 28 places, and not merely to 14 places. It is therefore likely that this 15-place value was computed by interpolation at a time when $\log 4799$ was not yet known.

Then, are logarithms involving smaller primes correctly given? For instance, are the logarithms of 119989, 119990, 119991 and 119992 correct? Well, again, $\log 119989 = \log(97 \cdot 1237)$ is given as 5.079141433895365 which is off by one unit of the last place. It may be explained by the fact that the logarithm of the factor 1237 was not yet known at that point. $\log 119990$ (largest factor 71) is correctly given, and so are $\log 119991$ (largest factor 47) and $\log 119992$ (largest factor 283). From this, we might conclude that the logarithms of primes at least up to 283 were used in the 15-place table.

But a little bit before, we find that $\log 119987 = \log(7 \cdot 61 \cdot 281)$ is given as 5.079134194930104 when it should be 5.079134194930103. This error might be explained by log 281 having not yet been computed. But on the next page, $\log 120012 = \log(2^2 \cdot 3 \cdot 73 \cdot 137)$ is given as 5.079224673324486 instead of the correct 5.079224673324487. This then raises some serious questions, and in particular how many of the logarithms of primes were really used, or how far they were used. I am not going to answer these questions, but a survey of the accuracy of the 547442 values involving only primes smaller than 10000 might enable us to pinpoint which logarithms were used and when.

5.2.2 What is the average accuracy?

I have not checked the entire range of values, but I have taken some samples. Errors are apparently very seldom larger than one unit of the 15th place. One such example is $\log 120025$ (largest factor 4801) which is given as 5.079271714641205 when the correct value is 5.079271714641203. Another is $\log 129980$ (largest factor 97) which is given as 5.113876532631054 instead of the correct 5.113876532631052. In this case, it is even more surprising, since the logarithms of all the factors were certainly known to Sang. This suggests that some logarithms of primes that had been computed were actually not used for the 15-place table.

The following range of the 15-place table shows that about half of the values are correct, and that the other half are off by one. I don't know if this is representative of the entire range 100000-150000.

```
127125: 5.104230965930801 (correct)
127126: 5.104234382196475 (correct)
127127: 5.104237798435276 (correct)
127128: 5.104241214647205 (off by 1)
127129: 5.104244630832261 (off by 1)
127130: 5.104248046990445 (off by 1)
127131: 5.104251463121758 (off by 1)
127132: 5.104254879226200 (correct)
```

```
127133: 5.104258295303772 (correct)
127134: 5.104261711354473 (correct)
127135: 5.104265127378305 (off by 1)
127136: 5.104268543375269 (correct)
127137: 5.104271959345364 (off by 1)
127138: 5.104275375288590 (off by 1)
127139: 5.104278791204948 (off by 1)
127140: 5.104282207094438 (correct)
127141: 5.104285622957062 (correct)
127142: 5.104289038792819 (correct)
127143: 5.104292454601710 (correct)
127144: 5.104295870383734 (off by 1)
127145: 5.104299286138894 (off by 1)
127146: 5.104302701867189 (correct)
127147: 5.104306117568620 (correct)
127148: 5.104309533243186 (correct)
127149: 5.104312948890888 (correct)
```

5.2.3 How was interpolation done?

We already know from the above that some gaps that could have been filled accurately were instead filled by interpolation and that the high accuracy of the logarithms of primes seems practically not used in the first range of the 15-place table.

We also know that Sang gave first and second differences, but these are all tabulated differences, that is, they are the exact differences of the rounded logarithms given, not the rounded actual exact differences.[8] In other words, the values in an interpolation cannot have been obtained by the differences in the tables, and these differences merely served to check the table. There was necessarily something more. For instance, more accurate values may have been used, and the second differences in the tables might then be rounded values of other values that we do not have access to.

Sang writes that the gaps in this range were filled by the interpolation of second differences [74, 3]. We can therefore assume that there were a certain number of unknown logarithms, surrounded by known logarithms. In the best case, there are three or four known logarithms on each side of the unknown sequence, but given that Sang did use interpolation much more than strictly necessary, we also have to consider the worst case, which is that of only one known logarithm on either side or the unknown sequence.

[8] This, then, is one of the differences with the *Cadastre* tables [36], where the differences given are really the differences which were used in the interpolations.

If interpolations are done with second differences, it is easy to see from the samples that it will be impossible to assume the second differences constant and at the same time expect errors of at most one unit on the interpolated logarithms. We have to take into account the fact that Δ^2 is not constant, but we may consider Δ^3 constant.

In general, the interpolation will start with a value of a logarithm, and values of the first, second and third differences. When Sang wrote that he used the interpolation of second differences, he may well have meant that the third differences were taken constant.

Now, the initial values of the differences can in certain cases be obtained from the values of the logarithms, but several consecutive values must be known. With two logarithms, we can compute the first difference, with three logarithms we can compute the second difference, and with four logarithms we can compute the third difference. Perhaps Sang did that in certain cases, and this is something that should be examined once the exact values of the 15-place have all been pinpointed. But in general, it seems that Sang would seldom have at hand four consecutive logarithms of numbers with factors say smaller than 100, and that he therefore seldom was in a position to compute the first three differences from the values of the logarithms. He may have been able to compute the first difference, sometimes the second difference, and only rarely the third difference this way.

Instead, I think it is much more likely that Sang supplemented the differences obtained from the computed logarithms with differences computed using the exact formulæ given earlier, namely $\Delta \log n$, $\Delta^2 \log n$, and $\Delta^3 \log n$.

This then raises two main questions: first, how accurate are the interpolated values using only Δ^3 and do we need more than 15 places to get at most one unit of error on the 15th place? second, what is the longest gap that can be filled that way?

I start by considering the gap of the logarithms 119981, 119982, and 119983 (figure 1). The two extreme values are prime, and 119982 has a prime factor greater than 10000. The values of $\log 119980$ and $\log 119984$ on the other hand can be computed from the logarithms of the primes under 10000. Even if Sang did not compute them exactly, I will make this assumption here, my purpose being to see how accurate the interpolated values can be when using only 15-place values. Sang's table has the values

```
119980: 5.079108857601436
119984: 5.079123336255966
```

which are correctly rounded. The values given by Sang in between are

```
119981: 5.079112477310321 (correct)
```

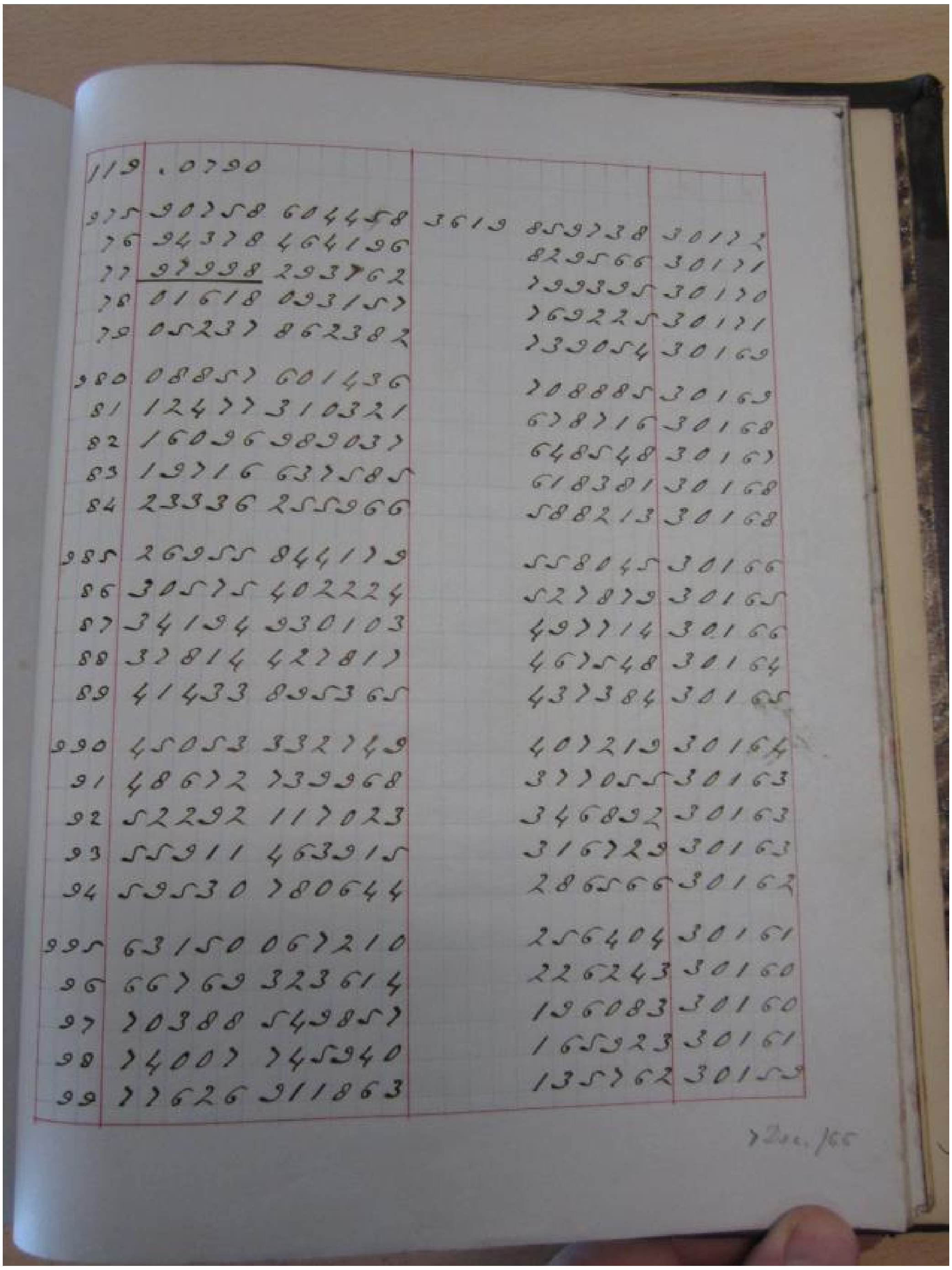

Figure 1: The logarithms from 119975 to 119999 (Photograph by the author, courtesy Edinburgh University Library). The date at the bottom reads 7 December 1866. The next page ends with the date 15 December 1866.

```
119982: 5.079116096989037 (should be 38)
119983: 5.079119716637585 (should be 86)
```

These values are not entirely correct and must have been interpolated.

Using the previously given formulæ, I add the first, second, and third differences. In order to make things more readable, I only show the last places up to the 15th:

```
119980: ...108857601436   3619708885 -30169  0.5
119981: ...112477310321   3619678716 -30168
119982:    116096989037   3619648548 -30168
119983:    119716637585   3619618380 -30167
119984:    123336255965
```

We can immediately see that the value of $\log 119984$ obtained here is off by only one unit of the 15th place, and in fact the values obtained for $\log 119981$, $\log 119982$, and $\log 119983$ are exactly those given by Sang. I may have been lucky, but in a certain way I have been working on 16 places, at least with respect to Δ^2. The initial value of Δ^2 was in fact about -30168.9, and I chose the second Δ^2 to be -30168 and not -30169. Had I done the opposite, I would have obtained

```
119980: ...108857601436   3619708885 -30169  0.5
119981: ...112477310321   3619678716 -30169
119982:    116096989037   3619648547 -30168
119983:    119716637584   3619618339 -30168
119984:    123336255963
```

and the logarithms of 119982 and 119983 would have been off by 2 units of the 15th place.

This seems to suggest that Sang has been using a constant Δ^3, and that he has been working on 16 places, at least for Δ^2 and Δ^3.

Let us now consider a longer interpolation, between $\log 119990 = \log(2 \cdot 5 \cdot 13^2 \cdot 71) = 5.0791450533327488 24\ldots$ and $\log 120000 = \log(2^6 \cdot 3 \cdot 5^4) = 5.079181246047624827\ldots$ which can both be computed with the logarithms of very small primes. Both logarithms are given correctly by Sang and were perhaps obtained that way. Like above, I scale everything by 10^{15} and I write only the last digits.

```
119990: ...145053332749  3619407219 -30164 0.5
119991:    148672739968  3619377055 -30163
119992:    152292117023  3619346892 -30163
```

```
119993:     155911463915  3619316729 -30162
119994:     159530780644  3619286567 -30162
119995:     163150067211  3619256405 -30161
119996:     166769323616  3619226244 -30161
119997:     170388549860  3619196083 -30160
119998:     174007745943  3619165923 -30160
119999:     177626911866  3619135763 -30159
120000:     181246047629
```

Here, the last value is off by 4 units of the 15th place. Of course, I have only been working on 15 places.

Let us now repeat this process but with only one more place for the logarithms and the first and second differences:

```
119990: ...1450533327488  36194072190 -301639 5
119991:     1486727399678  36193770551 -301634
119992:     1522921170229  36193468917 -301629
119993:     1559114639146  36193167288 -301624
119994:     1595307806434  36192865664 -301619
119995:     1631500672098  36192564045 -301614
119996:     1667693236143  36192262431 -301609
119997:     1703885498574  36191960822 -301604
119998:     1740077459396  36191659218 -301599
119999:     1776269118614  36191357619 -301594
120000:     1812460476233
```

Now, the error has been reduced to 2 units of the 15th place. The error could have been further reduced by working on 17 places. This is perhaps what Sang did, before rounding the values to 15 places. Here is that same interpolation, but on 17 places:

```
119990: ...14505333274882  361940721895 -3016386 50
119991:     14867273996777  361937705509 -3016336
119992:     15229211702286  361934689173 -3016286
119993:     15591146391459  361931672887 -3016236
119994:     15953078064346  361928656651 -3016186
119995:     16315006720997  361925640465 -3016136
119996:     16676932361462  361922624329 -3016086
119997:     17038854985791  361919608243 -3016036
119998:     17400774594034  361916592207 -3015986
119999:     17762691186241  361913576221 -3015936
120000:     18124604762462
```

If I round these values to 15 places, I now have the correct value of $\log 120000$. However, this is probably not what Sang did, since his values differ from mine (see figure 1). The cause of these discrepancies may lie in the use of different pivots.

5.2.4 Longest sequences

It is useful to have an idea of what are the longest sequences that may require interpolation. If all logarithms of primes smaller than 10000 are used, then the longest gaps are the two sequences of six integers:

$$
\begin{aligned}
109012 &= 2^2 \times 27253 \\
109013 &= 109013 \\
109014 &= 2 \times 3 \times 18169 \\
109015 &= 5 \times 21803 \\
109016 &= 2^3 \times 13627 \\
109017 &= 3^2 \times 12113
\end{aligned}
$$

$$
\begin{aligned}
128886 &= 2 \times 3 \times 21481 \\
128887 &= 11 \times 11717 \\
128888 &= 2^3 \times 16111 \\
128889 &= 3^2 \times 14321 \\
128890 &= 2 \times 5 \times 12889 \\
128891 &= 7 \times 18413
\end{aligned}
$$

In the first case, for instance, we can start with $\log 109011$, compute the first, second and third differences, and fill the gap. But if the threshhold is put smaller, then such sequences become longer. For instance, if we only use the logarithms of primes smaller than 1000, then the longest gap will be 14, for instance from $126921 = 3 \cdot 42307$ to $126934 = 2 \cdot 63467$. But if we put the threshhold at 100, that is if only the logarithms of primes smaller than 100 are used, then the longest gap to bridge will be 59 integers, for instance from $127101 = 3 \cdot 13 \cdot 3259$ to $127159 = 101 \cdot 1259$.

Could Sang have bridged such gaps with only constant third differences? We can in fact readily check it using Newton's interpolation formula.

Let us consider the last example, which is perhaps extreme, but insightful. We compute the logarithm of 127000 and the first, second and third differences. We settle on keeping the third difference constant, but we do not yet know on how many places we are going to work. We have:

$$\begin{aligned}\log 127000 &= 5.1038037209559568642469874218\\ \Delta^1 &= 0.0000034196281267041914740017\\ \Delta^2 &= -0.0000000000269258877000628103\\ \Delta^3 &= 0.0000000000000004240213806890\end{aligned}$$

with all values rounded to 28 places, although we will not need that many. As a comparison, I give the corresponding values at the end of the interval:

$$\begin{aligned}\log 127060 &= 5.1040088509992430158526873962\\ \Delta^1 &= 0.0000034180133236174385480814\\ \Delta^2 &= -0.0000000000269004641347121201\\ \Delta^3 &= 0.0000000000000004234209799922\end{aligned}$$

Let us now compute log 127060 using Newton's formula:

$$z_{60} = \log 127000 + 60\Delta^1 + 1770\Delta^2 + 34220\Delta^3$$

If now we want to have 15 correct places, the neglected terms of the interpolation formula should be at most about 1 unit of the 15th place. But in the above case, we readily see that Δ^3 carries a great weight, and in fact this term will not only affect the 11th place, but there will be an uncertainty on the 15th place if the value of Δ^3 is not extended at least to 20 places. In order to have a better grasp of the accuracy, we need to evaluate the neglected terms. Well, the neglected term in Newton's formula when considering Δ^3 constant is about

$$x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot \frac{x-3}{4} \Delta^4$$

and therefore, after l steps:

$$-M\frac{l(l-1)(l-2)(l-3)}{4 \cdot n^4}$$

With $n = 127000$ and $l = 60$, we obtain $0.00000\,00000\,00004\,88\ldots$.

In other words, and this is the important result, even if the values of the logarithm, the first, second and third differences are taken with 28 places (or more), we end up with an error of about 5 units of the 15th place.

The conclusion of this investigation is that if indeed Sang only used constant third differences, he can not have interpolated on such large intervals.

In order to keep the above uncertainty at about 1 unit of the 15th place, we first need to interpolate on intervals that are no greater than 40 values. If we go so far, then we have

$$z_{40} = \log 127000 + 40\Delta^1 + 780\Delta^2 + 9880\Delta^3$$

Taking again the above values, rounded to 20 places:

$$\begin{aligned}
\log 127000 &= 5.10380372095595686424698 74218\\
\Delta^1 &= 0.00000341962812670419147 40017\\
\Delta^2 &= -0.00000000002692588770006 28103\\
\Delta^3 &= 0.00000000000000004240213806890
\end{aligned}$$

we obtain log 127040 with an accuracy of 1 unit of the 15th place. This is still true if we start with values rounded to 18 or 19 places. But it is no longer true if we start with values rounded to 17 places. This is greatly due to the digits of Δ^3 then neglected. I conclude that with the use of constant third differences, Sang must have restricted the interpolation intervals to 40 values, and that he must have been working with initial values accurate to at least 18 places.

Keeping the interpolation intervals down to 40 values is quite feasible, as I mentioned above. For instance, if Sang would have only used the logarithms of primes smaller than 220, the longest gap would have been of 34 values. One such example is from 146081 to 146114. And even if Sang was facing a too wide interval, he may have broken it by either using some logarithm of prime which he would perhaps not have used otherwise, or he may even have done a special independent computation.

5.2.5 Conclusion

From the above, we can conclude that a number of logarithms in this range were computed from the logarithms of primes, but not all those that might have been, and sometimes not even those that should have been. It is also possible that there are some inconsistencies, that is that some logarithm was used here, but not there, and all this needs to be investigated.

We must assume that the logarithms computed from the logarithms of primes were correctly rounded. In between, other logarithms were interpolated. It does seem unlikely that Sang worked on only 15 places, at least in the range 100000-150000. I believe that Sang worked on 18 places or more, and that he computed the first, second and third differences to that accuracy, before eventually rounding the values to 15 places. Of course, if Sang proceeded that way, his computations bear a lot of similarities with those of Prony [36].

It seems that the 15-place values in the range 100000-150000 may be off by one unit of the last place, and in a few cases by two units, because for some reason some logarithms that could have been given accurately were not, one such example being 129980 whose largest factor is 97.

5.3 Range 150000-200000 (volumes K17-K21)

The values in this range were supposedly obtained from those in the previous range 100000-150000, where we have to assume that there might be one or two units of error in the last place.

Sang writes that he has computed two new logarithms between every pair of consecutive even integers, presumably in the range 100000-150000. In other words, starting with approximations of $\log(2n)$ and $\log(2n+2)$, Sang interpolated the values of $\log(2n+2/3)$ and $\log(2n+4/3)$, and adding $\log(3/2)$ to these values, he would obtain approximations of $\log(3n)$, $\log(3n+1)$, $\log(3n+2)$ and $\log(3n+3)$. 100000 would map to 150000 and 150000 to 225000. I do however not know if Sang used this process beyond 200000 and this is one of the matters that needs to be further investigated.

It also seems that Sang did not use the 28-place logarithms for filling the logarithms of numbers with factors no greater than 10000 in the range 150000-200000.

5.3.1 The accuracy of the values

In order to assess the accuracy of the values in this range, all the logarithm values between 150000 and 200000 should be checked and all deviations recorded. Of course, the differences need not be recorded, because those in the 15-place table have been tabulated and are derived from the logarithms and not the other way around.

I have only taken a few samples, and it is difficult to conclude effectively on the accuracy within the range 150000-200000. One might expect values which are somewhat less correct, as they were derived from a number of interpolated values in the range 100000-150000. Here are for instance the values given by Sang in the range 160000-160024:

```
160000: off by 1 unit (although its largest factor is 5)
160001: correct
160002: off by 1
160003: off by 1 (largest factor 61)
```

```
160004: off by 1 (largest factor 181)
160005: off by 1
160006: correct
160007: correct
160008: correct
160009: correct
160010: correct
160011: correct
160012: off by 1 (largest factor 367)
160013: off by 1
160014: correct
160015: correct
160016: off by 1 (largest factor 137)
160017: correct
160018: correct
160019: correct
160020: correct
160021: correct
160022: off by 1 (largest factor 89)
160023: off by 2 (largest factor 1301)
160024: off by 3 (largest factor 241)
160025: off by 3 (largest factor 179)
```

The large errors may have been due to interpolations from the range 100000-150000.

We see that except in a few cases, the logarithms are either correct or off by 1. The errors of 2 or 3 units seen above seem to be rather exceptions, but their cause will have to be investigated. The corresponding values around $\log 106682$ may be off by 2 units or more, causing the present discrepancies.

5.3.2 How accurate are the interpolations?

The errors mentionned in the previous section may have been due to errors in the original values in the first range. But what if the base values are all correct, or at most off by one unit of the 15th place? What can we expect with the new interpolations?

Let's insert for instance two logarithms between $\log 100114$ and $\log 100116$. Sang gives the values

```
100114: 5.000494813719108 (correct)
100115: 5.000499151696943 (off by 1)
```

```
100116: 5.000503489631449 (off by 2)
100117: 5.000507827522625 (off by 2)
100118: 5.000512165370472 (off by 2)
```

First, I am going to use only the first and second differences given by Sang and add them to my table:

```
100114: 5.000494813719108 0.000004337977835 -0.000000000043329
100115: 5.000499151696943
100116: 5.000503489631449
100117: 5.000507827522625
100118: 5.000512165370472
```

The two new logarithms are obtained with

$$z_{2/3} = \log 100114 + \frac{2\Delta^1}{3} - \frac{\Delta^2}{9} \tag{10}$$

$$z_{4/3} = \log 100114 + \frac{4\Delta^1}{3} + \frac{2\Delta^2}{9} \tag{11}$$

and we obtain

$$\log(100114 + 2/3) = 5.00049\,77057\,09145\,66666$$

$$\log(100114 + 4/3) = 5.00050\,05976\,79926\,00000$$

The results are correct to one unit of the 14th place.

More accurate values could have been obtained with the use of Δ^3.

But then, since Δ^3 can not accurately be obtained from the table, we need to compute it. For instance, in the previous case, we can use the previously given formula (8) for $\Delta^3 \log n$:

$$\Delta^3 \log 100114 \approx \frac{2M}{100114^3} = 0.000000000000000865\ldots$$

and therefore using

$$z_{2/3} = \log 100114 + \frac{2\Delta^1}{3} - \frac{\Delta^2}{9} + \frac{4\Delta^3}{81} \tag{12}$$

$$z_{4/3} = \log 100114 + \frac{4\Delta^1}{3} + \frac{2\Delta^2}{9} - \frac{4\Delta^3}{81} \tag{13}$$

we obtain

$$\log(100114 + 2/3) = 5.00049\ 77057\ 09145\ 70937$$
$$\log(100114 + 4/3) = 5.00050\ 05976\ 79925\ 95729$$

which are off by one unit of the 15th place.

In practice, however, Sang may have proceeded on longer runs, and computed the subtabulated differences. This means that instead of using formulæ (12) and (13), Sang may have started with the subtabulated differences

$$\begin{aligned}
\delta^1 &= 2\frac{\Delta^1}{3} - \frac{\Delta^2}{9} + \frac{4\Delta^3}{81},\\
\delta^2 &= \frac{4\Delta^2}{9} - \frac{12\Delta^3}{81}\\
\delta^3 &= \frac{24\Delta^3}{81}
\end{aligned}$$

and computed a number of values using differences. In fact, Sang writes that the interpolations were done using only the last two figures of the second differences [3], but we do not know clearly whether he used third differences, and when he computed these third differences.

In any case, once Sang had the values of the interpolated logarithms, he must have added $\log 1.5$ to each of these values. It is therefore possible to lose at most one unit of the 15th place when interpolating these two logarithms.

But then, if the interpolation is done between 100116 and 100118, whose values are off by 2 units, the two new values may also be off by 2 or more units. This suggests to investigate the values of $\log 150174$, $\log 150175$, $\log 150176$, and $\log 150177$ in the 15-place table, in order to see if the errors on $\log 100116$ and $\log 100118$ have been carried over to the range 150000-200000.

Now, if we accept that the values in the range 100000-150000 may be off by one unit of the 15th place, then we should also accept that the values in the range 150000-200000, or 150000-225000, are sometimes off by 2 units of the 15th place, if they have merely been obtained from the values in the first range.

5.4 Range 200000-400000 (volumes K22-K38)

In order to obtain the logarithms in the range 200000-400000 (figure 2), Sang writes that he has computed one new logarithm between every pair of consecutive integers, presumably in the range 100000-200000, or even 100000-225000. This allows him to map the range 100000-200000 on 200000-400000, or 200000-450000 if the second range were extended to 225000.

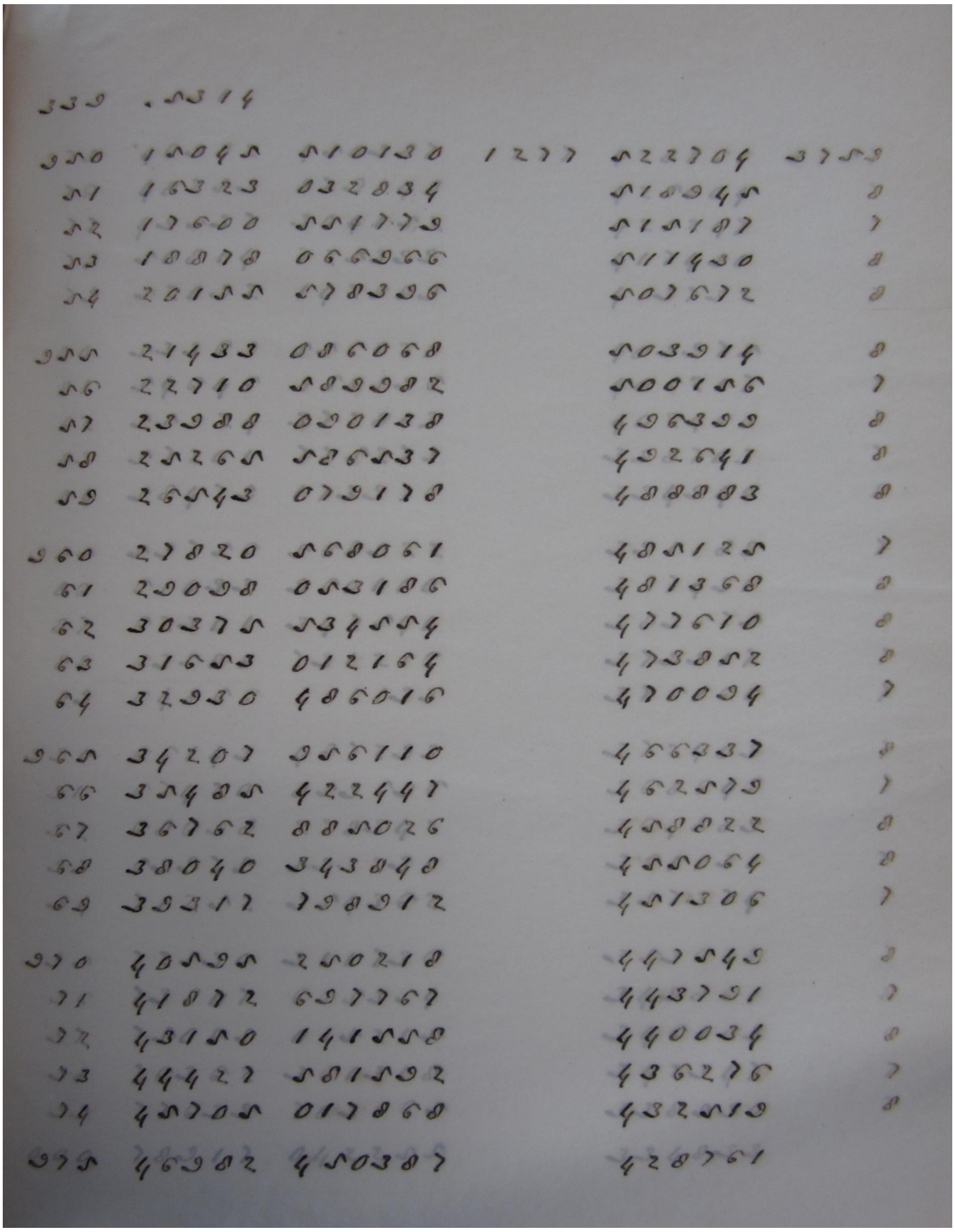

Figure 2: The 15-place logarithms from 339950 to 339975 (transfer copy, photograph by the author, courtesy Edinburgh University Library).

This raises some questions, for instance with the logarithms of 160023, 160024, 160025 which are already off by 2 or 3 units. Are the logarithms of 320046, 320047, 320048, 320049, and 320050 also off by 2, 3 or 4 units?

5.4.1 The accuracy of the values

The two samples below might suggest that the errors are kept most of the time at one unit over the entire range. In all the pages surveyed, I have only found a few cases with errors of 2 or 3 units in the 15th place.

In the range 205400 to 205425:

```
205400: correct
205401: off by 1
205402: correct
205403: correct
205404: correct
205405: correct
205406: correct
205407: correct
205408: correct
205409: correct
205410: correct
205411: correct
205412: off by 1 (largest factor 577)
205413: off by 1 (largest factor 229)
205414: off by 1
205415: correct
205416: correct
205417: off by 1
205418: off by 1 (largest factor 379)
205419: off by 1
205420: off by 1
205421: off by 1
205422: off by 1 (largest factor 73)
205423: off by 1
205424: off by 1 (largest factor 347)
205425: off by 1 (largest factor 83)
```

In the last range, from 369975 to 370000:

```
369975: correct
369976: correct
```

```
369977: correct
369978: correct
369979: off by 1
369980: correct
369981: correct
369982: off by 1
369983: correct
369984: off by 1 (largest factor 47)
369985: correct
369986: correct
369987: correct
369988: correct
369989: correct
369990: off by 1
369991: off by 1
369992: off by 1
369993: off by 1 (largest factor 179)
369994: off by 1
369995: correct
369996: correct
369997: off by 1
369998: off by 1
369999: off by 1
370000: off by 1 (largest factor 37)
```

But we can not be content with these two samples, as there are a number of issues with Sang's procedure. Errors in one range should be magnified in the next range. Perhaps I was just lucky with these samples and the actual accuracy of this range is not as good as it seems. Further study is needed!

5.4.2 The accuracy of the interpolations

We can nevertheless assume that the values in the range 100000-200000 are off by at most 2 units of the 15th place, and see what is the accuracy of the interpolated values.

Adding one value between two consecutive logarithms is done as follows:

$$z_{0.5} = \log n + \frac{\Delta^1}{2} - \frac{\Delta^2}{8} + \frac{\Delta^3}{16} \tag{14}$$

Let's take for instance $\log 100041$ and $\log 100042$, which Sang gives as

```
100041: 5.000178024245105 (off by 2)
100042: 5.000182365388349 (off by 1)
```

The first and second differences given are:

```
100041: 5.000178024245105 0.000004341143244 -0.000000000043393
100042: 5.000182365388349 0.000004341099851
```

From that, we obtain (with $\Delta^3 = 0$):

$\log 100041.5 = 5.00018019482215112500$ whose first 14 places are correct.

Again, a slightly more accurate value can be obtained if we compute the value of $\Delta^3 = .0000000000000008675\ldots$, but we can actually not do much better, given that the two base logarithms are off by 1 or 2 (in this case in the same direction).

The interpolations in this range were probably also done by using the last digits of the differences only, but Sang also writes that the results were checked by addition at least twice in each decade [3].

5.4.3 Conclusion

We should expect that some of the values in the range 100000-150000 are off by one unit of the 15th place, that some of the values in the range 150000-200000 or 150000-225000 are off by two units, and that some values of the range 200000-370000 are off by three units. This seems about the best that can be done. But for some reason, the actual accuracy is worse than that. I have mentioned that $\log 120025$ and $\log 129980$ are off by two units of the 15th place. There are probably other such cases, and these values may have influenced the interpolations in the two other ranges. However, as I mentioned before, I did not find any case of errors larger than 3 units of the 15th place in the range. These findings are therefore surprising and require further investigations.

5.5 Was there some tweaking?

Since Sang tabulated the first and second differences, it is possible that in certain cases discrepancies were noticed on the second differences. If such was the case, perhaps Sang slightly adjusted some values in order to smooth the deviations? This will be another area of investigation for future researchers.

5.6 How Sang might have reached the million

Let us now assume that the values in the range 100000-370000 are off by at most 3 units of the 15th place, which is what my samples seem to indicate. How could Sang have reached the million using these values, that is, how could he have filled the gap from 370000 to one million? Sang did not describe how he would be proceeding, and we are left to guessing it.

First, we should observe that with the previous procedure Sang could have reached 450000 with at most 3 units off on the 15th place, and we will assume this was the case. One way to reach the million would have been to insert two logarithms between each integer in the range 150000-333333 and to add $\log 3$ to each of them in. We would have the mapping

$$\begin{aligned}
\log n &\to \log(3n)\\
\log(n+1/3) &\to \log(3n+1)\\
\log(n+2/3) &\to \log(3n+2)\\
\log(n+1) &\to \log(3n+3)
\end{aligned}$$

The question is therefore to see what accuracy can be obtained by adding two values between logarithms which may be off by up to three units of the 15th place. For instance, let us add two values between the logarithms of 160000 and 160001, taking the values (logarithms and differences) found in the 15-place table, $\log 160000$ being actually off by one unit:

```
160000:      5.204119982655924    0.000002714332030   -0.000000000016965
160000+1/3:
160000+2/3:
160001:      5.204122696987954
```

We can use Newton's formula (4) which, when setting $\Delta^4 = 0$, reduces to:

$$z_x = A + x\Delta^1 + x \cdot \frac{x-1}{2}\Delta^2 + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3}\Delta^3 \tag{15}$$

Setting even further $\Delta^3 = 0$, that is, using only the tabulated values in the range 150000-333333, we obtain:

$$z_{1/3} = \log 160000 + \frac{\Delta^1}{3} - \frac{\Delta^2}{9} \tag{16}$$

$$z_{2/3} = \log 160000 + \frac{2\Delta^1}{3} - \frac{\Delta^2}{9} \tag{17}$$

resulting in

$$\log(160000 + 1/3) = 5.20412\ 08874\ 35152\ 33332$$
$$\log(160000 + 2/3) = 5.20412\ 17922\ 12495\ 66665$$

These values are off by at most one unit of the 15th place. Slightly more accurate values could have been obtained using Δ^3, but in fact Sang could have dispensed with it and still obtained the values of the logarithms up to one million off by at most four units of the 15th place.

5.7 Another way to compute the 15-place table

As I have stressed above, Sang did not make use of the logarithms of primes as much as he could. It seems that in some cases even logarithms of small primes such as 97 were not systematically used in the range 100000-150000, and perhaps not at all beyond 150000. This may have been the case in order to save time, but if one had the time, or an army of computers at hand, it would have been more interesting to use all the base logarithms computed, perhaps also in view of going even further than a 9-place table in the future.

With this approach, the logarithms of primes would have been used as much as possible and only non interpolated values would have been used for interpolations, using computed first, second and third differences.

In that case, the known logarithms used for interpolation could be considered correctly rounded, and the question is then what is the longest gap to bridge, this time on the entire range 100000-1000000.

This longest gap happens to be of length 12 and spans from $\log 911964$ to $\log 911975$:

$$911963 = 13 \times 29 \times 41 \times 59$$
$$911964 = 2^2 \times 3 \times 75997$$
$$911965 = 5 \times 17 \times 10729$$
$$911966 = 2 \times 11 \times 41453$$
$$911967 = 3 \times 7 \times 43427$$
$$911968 = 2^5 \times 28499$$
$$911969 = 911969$$
$$911970 = 2 \times 3^2 \times 5 \times 10133$$
$$911971 = 53 \times 17207$$
$$911972 = 2^2 \times 227993$$
$$911973 = 3 \times 23 \times 13217$$
$$911974 = 2 \times 7 \times 65141$$
$$911975 = 5^2 \times 36479$$
$$911976 = 2^3 \times 3 \times 13 \times 37 \times 79$$

We could then assume that the values of $\log 911963$ and $\log 911976$ are off by at most one unit of the 27th place, and using an interpolation with constant third differences, this would result in in an error of about

$$-M\frac{l(l-1)(l-2)(l-3)}{4 \cdot n^4}$$

with $l = 12$ and $n = 911963$, hence about 2 units of the 21th place. On the other end of the range, around 100000, such an interpolation would produce an error of about one unit of the 17th place. But if higher differences are used, it would be possible to obtain the intermediate values to almost 27 places, given that the surrounding values might be off by one unit of the 27th place.

6 Reduced tables

Two reduced tables of logarithms were to be derived from the 15-place table, a 7-place table and a 9-place table.

6.1 The 7-place table (1871)

This table gives the logarithms to 7 places from 20000 to 200000 and was published in 1871 [61]. It was extracted from the 15-place table which covers the range 10000 to 370000. In fact, we know from the specimens Sang published in 1872 [64] that he had then computed the 15-place table up to 260000 and also the 28-place primes up to 2600.

The 7-place table was easily derived, as no recomputations were needed because of last-place uncertainties.

I have reconstructed this 7-place table [33].

6.2 The 9-place table (1872 project)

Sang had planned to publish a 9-place table of logarithms, which I have (re)constructed [31]. This was Sang's main project, of which the 7-place table was a byproduct. The 9-place table would have covered the range from 100000 to one million, but could have been extended from 1 to one million, since the range from 1 to 100000 can be derived almost immediately from the range 100000-1000000.

The main part of this table would have been extracted from the 15-place tables. Sang chose to have a 15 place table, so that on average each sequence of six digits from the 10th to the 15th place would only occur once in a table from 100000 to one million. This means that there would be only a few sequences such as 499997, 499998, 499999, 500000, 500001, 500002, or 500003, and that almost every logarithm to 15 place could immediately be turned into a 9-place logarithm. For the few border cases, a special computation can be done, that is, the logarithm of these numbers can be computed directly. However, if the 15-place table were replaced by a table to 14 places or less, there would be at least 10 times more logarithms to recompute. There is therefore a trade-off.

Of course, in order to apply this procedure to the 9-place table, Sang would have had to extend the 15-place table to one million which currently only reaches 370000.

7 Need for further work

All of the above observations enable us to draw a clearer picture of Sang's work, and also to trace the paths of future investigations.

Right now, we need to find out exactly which ones of Sang's 270000 values are correct, which ones are off by 1, which ones are off by 2, and so on. Once the exact deviations (0, +1, −1, +2, −2, etc.) are recorded, we should identify the logarithms which have been computed from the logarithms of primes. We can start with the assumption that whenever some logarithm of prime was used, it was used in all the places where this prime is involved.

Next, we have to reconstruct the interpolations and compare them with the values found by Sang.

In order to locate the pivots on the range 100000-150000, we have to keep only those correct values whose largest factor is not the largest factor of another logarithm whose value is not correct. For if this is the case, it would likely mean that the first value is only accidentally true and that the logarithm of this factor was not used in the computations of the pivots.

Once these identifications are done, interpolations may enable us to pinpoint a number of integers which have been subject to special calculations, for instance in order to reduce the range of the interpolations.

It may be, however, that things are more complex, and that certain logarithms of primes were only used for certain pivots. Once the range 100000-150000 is analyzed and the exact calculations of Sang have been identified, we can turn our attention to the other two ranges. Eventually, it will be essential to analyze how errors in one range transfer to the other ranges. From my samples, it would seem that the values in the range 200000-370000 are off by at most 3 units of the 15th place. Is this correct? If large errors in earlier ranges were not transfered to later ranges, how were the values corrected? These are some of many questions which are awaiting the future researchers.

8 References

The following list covers the most important references[9] related to Sang's table. Not all items of this list are mentioned in the text, and the sources which have not been seen are marked so. I have added notes about the contents of the articles in certain cases.

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[6] Henry Briggs. *Arithmetica logarithmica.* London: William Jones, 1624. [The tables were reconstructed by D. Roegel in 2010. [34]].

[9]**Note on the titles of the works:** Original titles come with many idiosyncrasies and features (line splitting, size, fonts, etc.) which can often not be reproduced in a list of references. It has therefore seemed pointless to capitalize works according to conventions which not only have no relation with the original work, but also do not restore the title entirely. In the following list of references, most title words (except in German) will therefore be left uncapitalized. The names of the authors have also been homogenized and initials expanded, as much as possible.

The reader should keep in mind that this list is not meant as a facsimile of the original works. The original style information could no doubt have been added as a note, but I have not done it here.

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