What did Napier invent?

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The tercentenary of the publication of Napier's *Descriptio* took place in 1914. At that time, logarithms were everywhere and new tables were still being constructed by hand. It will suffice to mention the great tables of Bauschinger-Peters (1910–1911) and Andoyer (1911). Machine-assisted computations made their debuts, but few were able to anticipate the dramatic changes that the 20th century would bring in computation.

But for all these groundbreaking efforts, the vast majority of the tables of logarithms were rooted several centuries before. Most of the tables then in use were not newly computed tables, but tables collated from other tables, carefully checked and compared, and which eventually derived from the first large tables of logarithms computed by Briggs (1624, 1633) and Vlacq (1628, 1633).

In 2014, tables of logarithms have long fallen into oblivion, although the digital age has made it possible to resurrect many of them. Anybody who wishes to use a table of logarithms can now print one and get his/her hands on the calculation methods which were commonplace for engineers, accountants and other professions until the 1970s. This is not that far away.

Nowadays, tables of logarithms are of course no longer needed. They have been replaced by handheld calculators, and even by the virtual calculators of our smartphones. Logarithms were used to simplify multiplications and divisions, but we can now multiply and divide without logarithms. It would therefore seem that commemorating Napier's work is now a bit off-topic. But is it?

In fact, the logarithmic function is alive and well. True, tables of logarithms are gone, but logarithms and exponentials appear everywhere as soon as one goes a little bit beyond elementary computations. Logarithms are not a function of the past, but the function has just become easier to compute than in the past.

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Still, there is a veil of confusion surrounding the invention of logarithms. In order to get a better understanding of what were Napier's innovations, we need to consider the context. Napier did not appear out of nothing, and it is important to examine the roots of his work, and also how it compares to similar and contemporary work. It is interesting to ponder the topics of the tercentenary and quatercentenary meetings. In 1914, we had communications on the invention of logarithms, on Napier, on the computation of logarithms, on the law of exponents, on alleged prior inventions of logarithms, on the change to Briggs's logarithms, on Napierian logarithms calculated before Napier, on Edward Sang's tables, on fundamental trigonometrical and logarithmic tables, and on a number of other subjects [8]. Topics in 2014 included the methods used by Napier, Napier's other calculating devices, how tables of logarithms pervaded Europe and beyond, their publication in Austria, Babbage's table of logarithms, Bauschinger and Peters's table published shortly before the 1914 meeting, and the work of Scottish mathematicians on logarithms. These are all important topics, showing the many ramifications of an apparently simple object.

What exactly happened around 1614 also needs a careful examination for the very reason that not all was new. This does explain why there have been claims for prior inventions of logarithms, sometimes attributed to Jost Bürgi (1552–1632), to Michael Stifel (1487–1657), or even to Archimedes (3rd century BC). Such priority claims are not new, and some of them had already been considered in 1914. But if we hope to settle the matter and make everybody happy, we ought of course to be accurate in our wordings and be clear not only about what are "logarithms," but also about what it means to be the inventor of logarithms. We cannot answer such questions without first defining what we mean by these terms. For instance, if we focus on the law of exponents ($a^n \cdot a^m = a^{n+m}$), then the invention of logarithms has been brought back by some to Archimedes. But should we focus only on the law of exponents?

Before trying to define what we understand by "logarithm," let us do a little historical sketch and consider a number of significant early developments related to logarithms.

The law of exponents

One of the earliest expressions of the law of exponents actually even goes beyond Archimedes (c287 BC-c212 BC), since it is found in Euclid's writings, slightly before Archimedes. Euclid flourished around 300 BC, and in book IX of the *Elements*, he states the following proposition:

> "*If numbers in any amount are continuously in proportion from the unit, the smallest one measures the largest one according to one of*

> *those among the numbers in proportion*." (our translation from [3, p. 424])

Heath translated it as follows:

> "*If as many numbers as we please beginning from an unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers*."[6, p. 395]

This may appear somewhat opaque, but considering that "measuring" is taken in the sense that "2 cm measures 10 cm five times," all what Euclid means is that $a^n/a = a^i$ with $1 \leq i \leq n$. Euclid adds the porism:

> "*And it is obvious that the place that the measuring number has from the unit is the same as the one that the number according to which we measure, has from the measured number, towards the preceding one*." (our translation from [3, pp. 424–425])

that Heath translated as

> "*And it is manifest that, whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.*" [6, p. 396]

This is to be understood as follows. Let a^r be the "measuring number" at place r. This number measures another number a^n which is at place n. At place $n-r$, there is the number which measures equally a^n. In other words, $a^r = a^n/a^{n-r}$, or in modern terms $a^n = a^r \cdot a^{n-r}$.

The law of exponents may have been buried in words, but it was known to Euclid. In fact it was certainly pretty obvious when considering a sequence of numbers in proportion.

Archimedes was somewhat more explicit. In the *Sand-reckoner*, a treatise on the definition and use of very large numbers, he wrote the following:

> "*If numbers are in continuous proportion from the unit and if some of them are multiplied together, the product will, in the same progression, be removed from the greatest of the multiplied numbers by as many numbers as the smallest multiplied number is removed from unit in that progression, and removed from unit by the sum minus one of the numbers of which the multiplied numbers are removed from the unit.*" (our translation from [17, p. 366])

In 1897, Heath [5, pp. 229–230] gave a more modernized translation:

> "*If there be any number of terms of a series in continued proportion, say* A_1, A_2, A_3, $\ldots$ A_m, $\ldots$ A_n, $\ldots$ A_{m+n-1}, $\ldots$ *of which* $A_1 = 1$, $A_2 = 10$, *and if any two terms as* A_m, A_n *be taken and multiplied, the product* $A_m \cdot A_n$ *will be a term in the same series and will be as many terms distant from* A_n *as* A_m *is distant from* A_1*; also it will be distant from* A_1 *by a number of terms less by one than the sum of the numbers of terms by which* A_m *and* A_n *respectively are distant from* A_1."

In other words, in this passage, Archimedes considers a sequence of numbers a, b, c, etc., such that the ratio between a and b is the same as between b and c, and so on. a is assumed to be equal to 1. Archimedes also uses a notion of distance from the unit, a being at distance 1 from a, b at distance 2, c at distance 3, and so forth. In modern terms, given the sequence $a_1 = 1$, a_2, a_3, $\ldots$, with $a_{i+1} = ra_i$, Archimedes merely says that if $a_k = a_i \times a_j$, and $i < j$, then $k - j + 1 = i$ and $k = i + j - 1$. If we write $a_i = r^{i-1}$, Archimedes in fact expresses that $r^{i-1} \times r^{j-1} = r^{(i+j-1)-1}$. This is indeed equivalent to the law of exponents, but Archimedes only uses it as a way to bridge the sequences of large numbers he has introduced.

We can see that Archimedes was manipulating simultaneously ratios and distances, or ratios and indices, and he noticed a simple property of indices. However noticing this property of indices does not mean that the indices were thought to be "functions" of the numbers. They were much more functions of the sequences.

A great breakthrough in the law of exponents seems to have been made by Nicole Oresme (c1320–1382). In his *Algorismus proportionum*, Oresme of course knows about the law of exponents with integers, but he also does come up with fractional exponents. In one of his examples, he shows that $(2/1)^{1/2} \times (3/1)^{1/3}$ is equal to $(72/1)^{1/6}$ [4, p. 338]. One should however be cautious and not hastily conclude that Oresme's notion of ratios was identical with the modern one, as recently highlighted by Rommevaux [15, p. 32].

The 15th century then witnessed the first "tables" of correspondences between arithmetic and integer sequences. Nicolas Chuquet (c1440–c1500), for instance, in his *Le triparty en la science des nombres* (c1484), considers the sequence 1, 2, 4, 8, 16, etc., and puts it in correspondence with the integers 0, 1, 2, etc. Chuquet then shows how the "denominations" are added when the numbers of the first sequence are multiplied [9, pp. 155–156].

Half a century later, it was the turn of Michael Stifel (c1486–1567). In his *Arithmetica integra* [16, folio 249], published in 1544, Stifel shows a correspondence between the integers from -3 to 6 and the corresponding powers of 2. In order to multiply $1/8$ by 64, Stifel says that the indices (-3 and 6) can be added, yielding 3, and the answer is therefore the power of 2 corresponding to the index 3, namely 8. The use of negative integers allows Stifel to deal with powers of 2 smaller than unity, but Stifel did not use fractional exponents.

The law of exponents, in various forms, was no doubt rediscovered by many others. Mention should be made of the Algerian Ibn Hamza who published in 1591 a treatise of arithmetic in which he basically expressed the law of exponents. This would be a footnote, except that a commentary published in 1913 interpreted Ibn Hamza's discovery as a capacity to discover logarithms, and this in turn generated a vast literature [1].

Anticipating logarithms

Logarithms have a strong connection with powers and the law of exponents, and a notion of logarithms was therefore latent in any work making use of this law. But in addition to the works displaying a knowledge of the law of exponents, there have also been cases where the solution to some problems could have benefited from logarithms.

In his *Problèmes numériques faisant suite et servant d'application au Triparty en la Science des nombres*, Chuquet does for instance consider a problem in which a barrel of wine loses a tenth of its content each day [10, p. 29, problem XCIV]. Chuquet wants to find out when the barrel will be half empty. This is a problem where logarithms prove handy, but Chuquet didn't have them. Chuquet's first answer to his problem is 6 days and $31441/531441$ fractions of a day. Given that $0.9^6 = 0.531441$ and $0.9^7 = 0.4782969$, Chuquet's answer is perhaps a typo for the linear interpolation $31441/(531441-478297)$, an approximation of $314410/(5314410-4782969) = 0.591617\ldots$. In any case, Chuquet says that many will be happy with this answer, but that the real value is a certain number still unknown. We can therefore presume that Chuquet had an understanding of the inadequacy of a mere linear interpolation. The correct answer is $6.578813\ldots$ days, and the linear interpolation is in fact not that bad.

Luca Pacioli (c1445–1517) on the other hand, considered a similar problem, but provided a solution which nowadays would be obtained using logarithms. In his *Summa de Arithmetica* published in 1494 [12], he considered a capital increasing with an annual rate of r percent, and he wanted to know how long it would take to double that capital. Pacioli wrote that the number of years for doubling the capital is obtained by dividing 72 by the annual rate [8, p. 163], [12, folio 181]. Assuming the rate is small, the number of years can indeed by approximated by $\ln(2)\times 100/r = 69.3/r$, which is not too far from $72/r$. In fact, Pacioli must have found that when the rate is small, there is almost an inverse linear relationship between the rate and the number of years. This result can be established knowing the properties of logarithms, but that is of course very far from sufficient to conclude that Pacioli anticipated logarithms.

The origins of functions

Nowadays, the logarithm is viewed as a *function*. It is an object which takes a value and yields another one. The function establishes a correspondence between two sets. It is therefore important to have some idea about the emergence of this notion.

Among the earliest manifestations of functions were certainly the cases where curves were thought to be "generated" by some kind of motion, so-called kinematic curves. To different times corresponded different positions. This was true for celestial motions, but also for some mathematical curves, such as Hippias's trisectrix used around 420 BC [2, p. 56]. In that curve, two points are defined as having uniform motions, one along a segment, another along an arc, and at each moment these two points are in a configuration which is used to define a third point, the one which is part of the new curve. Alternatively, one can also consider that one segment has a uniform translation motion, and that the other rotates uniformly. Not only is there a correspondence between a time and a new position, but also between three points throughout time.

When Napier defined his logarithms, he also created two motions, one uniform, the other not, but such that two sequences of points were defined, and such that these points could be put in correspondence. Like in the quadratrix, the correspondence gave birth to a function. And these functions were implicitely continuous, each point going through all the positions of a given curve [11, pp. 82–83]. Napier's kinematic construction, then, should also be put in the context of the Oxford Calculators of the 14th century, who used such techniques [18].

Continuity

These considerations naturally lead to the important notion of continuity, which is essential for the modern notion of logarithms. If we return to the consideration of powers and ratios, we can see that at the beginning, exponents were discrete. Ratios were applied as a whole. Exponents were discrete just as a sequence was considered as something being countable, having a first element, then a second one, a third one, and so on. This led to gaps, and at the same time, whenever a correspondence was defined, such as those between integers and powers, this raised the question of what happened in between. If 2 corresponds to 4 and 3 to 8, to what does 2 and a half correspond? Such questions may be irrelevant if one works only with integers, but if one considers that the integers represent time, or even a position, then intermediate values have a clear meaning, and the question makes sense.

The underlying notion of continuity was probably almost intuitive, as it is so closely connected with that of motion. When an object moves from one place to another, it is natural to consider the transition as smooth and to assume that the object has occupied every intermediate position. But from an abstract and more mathematical point of view, continuity was not something totally obvious. It seemed intuitive, but defining it was not so easy. Moreover, there have been competing views. For Aristotle, for instance, something continuous was something that could be divided infinitely, a never-ending division process. A continuous line could therefore not be made of indivisible points. For Archimedes, on the other hand, continuity meant to be made of an infinity of indivisible elements. For instance, a plane can be thought as made of an infinity of lines, and a line as an infinity of points. Archimedes's approach is much closer to the modern one than Aristotle's.

The first practical tables

Napier and Bürgi are the authors of the first practical tables for simplifying calculations based on the replacement of multiplications by additions. The theory underlying both tables is the same. Bürgi's system appears much simpler and much easier to understand than Napier's. The very construction of Bürgi's table is so straightforward that practically anybody could recompute his table, although it would take some time. Earlier authors had some of Napier and Bürgi's ideas, but they did not construct extensive tables, and don't seem to have contemplated doing so. Now, what distinguishes Napier and Bürgi?

Bürgi

Jost Bürgi is a lesser known figure than Napier, in particular because no mathematical concept bears his name. But Bürgi was a very skilled mechanician, an astronomer, as well as a mathematician. A recently rediscovered trigonometry manuscript from the 1580s shows how much Bürgi was ahead of his time. That he constructed a table to simplify multiplications is therefore not a surprise. But how does this table compare to Napier's?

Whereas Stifel used powers of 2, and did not provide an extensive table, Bürgi understood that to make the process useable, all of the numbers should be reachable, and that this could be obtained to some extent by taking the powers of a number close to 1. By filling the interval 10^8 to 10^9 (or between 1 and 10 for simplification) using powers of a number close to 1, Bürgi provided a means to replace multiplications by additions, as an extension of Stifel's scheme. Bürgi neither invented the law of exponents, nor fractional exponents, but to a number of integers, he associated values between 1 and 10, and he extended this correspondence to several non integer indices between 230270 and 230270.022 in order to close the gap with 10. This necessarily meant that 0.022 was a measure of a (multiplicative) fraction of the ratio 1.0001, the one needed to go from 999999779 to 10^9. It doesn't matter that much at this point as to how Bürgi computed this association, be it through an interpolation, or differently. But he had $1 \to 1.0001$ and $0.022 \to \frac{10^9}{999999779}$. This of course denotes some kind of understanding of fractional powers.

Admittedly, Bürgi necessarily also had the converse understanding, that associating an index to a number, because this is exactly how his table was used: a value was looked up, and its index was found, and used in an addition. So, in a way, Bürgi's table is used to find the logarithm of a number, and later the antilogarithm of another.

But this idea of associating an index to a value and conversely can not be ascribed to Bürgi. Stifel and others had this understanding, at least for the values of their discrete sequences, and Oresme, even earlier, had an understanding that some ratio was obtainable as the power of another ratio. Still, Bürgi was one of those who made this idea useable for ordinary calculations.

Napier's abstract definition of a function

Napier on the other hand started with an abstract definition of a relation, which can be viewed as a specification. He then wanted to take measurements on the motions that he had specified. And he needed continuity and interpolation methods. Each of his two motions were made dependent on time (or at least on a sequence of integers meant to represent uniform time), but the relationship between the two motions did not depend on time anymore. It only depended on the positions. Time was only a tool used for constructing a relationship. This is like constructing a square in order to measure the ratio of its diagonal to its side. The dimensions of the square have to be specified, but the ratio will be independent of these dimensions.

Napier's purpose was chiefly to replace ratios by differences, and to provide a table for that effect. His aim was more specialized than Bürgi's, in that he was more interested in trigonometrical calculations than into mere multiplications. He realized that it was possible to combine tables of (pure) logarithms and trigonometrical tables. Napier was certainly right, and facilitating trigonometrical calculations was badly needed, but "pure" multiplications were also useful, and in fact the first table of pure (non trigonometrical) logarithms appeared in 1617, only three years after Napier's *Descriptio*.

Interval arithmetic and the origins of accuracy

Napier's construction was admittedly complex. He defined several sequences of numbers, using different ratios, and he needed to associate these numbers n to others $L(n)$, which are their logarithms. Given the complex structure of his sequences, there was a strong need to be able to estimate the accuracy of the values Napier obtained, as an incorrect value would not be conspicuously wrong.

In order to control the accuracy of his logarithms, Napier made perhaps the first use of interval arithmetic. Instead of manipulating mere values, Napier did manipulate pairs of values. Taking a simple example, $\sqrt{2}$ could be represented as the pair $(1.41, 1.42)$, and then one might deduce that $2\sqrt{2}$ can be represented by $(2.82, 2.84)$. Interval arithmetic was made easier by the geometric series employed by Napier. In his definitions, Napier took $L(10^7) = 0$ and he found that $1 < L(9999999) < 1.00000010000001...$ Napier chose the average of the two surrounding values as an approximation of the real value. He knew therefore that the error of the logarithm was less than 10^{-7} and presumed that the logarithm was close to $1 + 5 \cdot 10^{-8}$. The actual error on this value is in fact smaller than 10^{-14}, but Napier didn't know it. Napier would have had his first logarithm equal to 1 if he had taken a slightly larger ratio than 0.9999999 in his first sequence, but this would have been extremely impractical. In any case, once Napier had an approximation of the first logarithm, he used it to obtain interval approximations of the next values in his sequences.

Napier's purpose was that his three construction (base) tables be as accurate as possible. It was on these tables that the actual canon was based. Interval arithmetic was actually confined to the construction, but could have been extended to the computation of actual logarithms beyond those of the construction tables. However, in order to use interval arithmetic on the values in the canon, something should have been known on the accuracy of the sine values.

Napier knew that a given logarithm was precisely defined by his kinematic construction. Bürgi, instead, adopted a different construction. He too could ensure the accuracy of his table, by various checks. Using various multiplications, Bürgi was certainly able to avoid the propagation of errors, and hence to bound his errors. That may also have been done by Napier, but only on his numbers, not on his logarithms. Bürgi only had integer indexes, and therefore didn't need to approximate their values.

Since Bürgi started with integer indices (by increments of 10), it may not have been totally clear to which index a given number was associated. All we know is that some interpolation may have been used to find which index produces 10 (or 10^9). For instance, given the values for 4000 and 4010, to what number does index 4001 correspond? How does Bürgi answer that question? If we do a linear interpolation between the values for 4000 (104080869) and 4010 (104091277), we obtain 104081909.80... An exact calculation gives 104081910.03... which is pretty good. But the accuracy depends on what values one uses, and there is no expression of that accuracy.

Of course, these considerations of accuracy could be taken into account in Bürgi's table, but they are nowhere apparent in the manual of the table and there does therefore appear a clear gap between the groundbreaking work of Napier, and Bürgi's extension of earlier works. This distinction was already made twenty years ago by Whiteside, and recently published in a posthumous article [18]. It is as if Napier was climbing Everest, making several simultaneous breakthroughs, like climbing several sides: the abstract definition of a function, the separate implementation of this function, a high concern for the accuracy of his fundamental table, an early use of interval arithmetic, a fundamental table which could be used for arithmetic calculations, and an applied trigonometrical table built upon the fundamental table. Bürgi's table on the other hand serves the same purpose as Napier's construction tables, but anticipated the pure arithmetic tables of logarithms of which Briggs' 1617 table was the first vanguard.

Defining the invention

Now that we have a better understanding of each contribution, we may attempt to define properly what it could mean to have invented logarithms. Of course, such a definition cannot purely focus on the name coined by Napier. Inventing logarithms also cannot merely be about the law of exponents, or else Euclid, Archimedes, and others will come first, and we will be in trouble defining what Napier (or Bürgi) did.

Bürgi and Napier both introduced *tables* for simplifying calculations. But we feel that tables are not necessary for defining the invention of logarithms, as one could obviously write down their theory without providing tables. Tables certainly do complement a theoretical basis. One could also provide a table, without a firm theoretical basis, as Bürgi did. Sometimes, a theoretical basis may have existed, but if it can't be produced, we cannot assume that it did exist. In the case of Bürgi, constructing his table is rather easy, and a theoretical basis is hardly needed.

One problem that needs to be examined is how Bürgi's work departs from those that came before, such as Stifel's work. Stifel and others showed how the indices in a progression could be used to multiply two terms. Bürgi's innovation was to make this property practical. He did so by using a ratio very close to 1, and at the same time he divided the interval [1,10] so that every number could be located in this interval. In a way, of course, Bürgi is even closer than Napier to modern tables of logarithms. Bürgi already had 10 play a fundamental role, and he had a means to obtain the index of any number, possibly using interpolation.

But Bürgi's correspondence also had problems. Different numbers had the same index. For instance, 64570 is the index of 190726011, but also of 1.90726011, 19.0726011, etc. This, admittedly, is also true of modern tables of logarithms, where one would look up in the same place, whether one is searching for $\log 11$, $\log 110$, or $\log 1100$. Napier, instead, really defined a function having different values for different arguments. There were not two numbers having the same logarithm. Napier's table was much closer to the theoretical logarithm than was Bürgi's table.

However Napier went beyond, or rather, he started with an abstract view of logarithms, which he tried to make practical. This step is totally new and it anticipates the development of calculus. Without a proper notation for functions, Napier defined a function, and constructed a procedure to evaluate the values of this function, in view of constructing a table of fundamental values, itself in order to use it for a table of logarithmic sines. This, and also Napier's interval arithmetic, does not seem to have any equivalent in Bürgi's work, even though Bürgi's tables were very accurate. Bürgi may have produced a much simpler table, and he did in some way anticipate modern tables of logarithms, but he did not display a grasp of an abstract function, in particular because he did not need any.

And this is exactly what we believe defines the invention of logarithms. Logarithms did no longer appear backstage, they did not have a mere implicit appearance any longer. For the first time in 1614, the function was put in the front and from its properties, tables were derived, using an array of both ancient and new techniques.

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