DIFFERENCE ENGINES IN THE 20th CENTURY

Stephan Weiss

Graduated from the TH Vienna and the TU Munich with a M.Sci. (Dipl. Ing.) in Mechanical Engineering. After further studies working as an expert in the fields of vehicles until 2007. Privately my special interests apply to the development of a collection of calculating machines and devices as well as handling of freely selected topics of the history of computing aids.
Website: http://www.mechrech.info email: ste@mechrech.info

Introduction

Related to difference engines and their inventors first of all the names Babbage, Scheutz, Wiberg or Grant come in mind. During the 19th century the named men built machines of that type or at least made some attempts. Their history and success or failure has been often documented and is well known. With the end of 19th century the history of difference engines didn't come to an end. During the next century two innovative new difference engines were built and used for calculating logarithmic tables. In scientific papers they are incidentally named and if, only little information is given. This article will throw some more lights on these difference engines of 20th century.
Our first question is what is a difference engine and what is it used for?

What is a difference engine?

A difference engine is a historical, mechanical special-purpose calculating machine designed to tabulate polynomial functions. Our next question is how does it work and what is it good for? This question can be answered best with an example. Assuming we have to tabulate the function $f(x) = x^2 + 2x - 7$

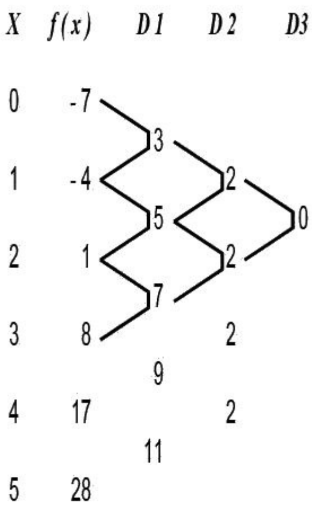


Figure 1 - function $f(x) = x^2 + 2x - 7$ and its differences

First we calculate some function values *f(x)*, then the first differences *D1* between successive values and finally the second differences *D2* between the first differences which become constant (fig. 1). With a polynomial of order n ($f(x) = x^n + ...$) the differences *Dn* become constant and all differences of higher order are therefore zero. By inspection of fig. 1 it is obvious that all following values of *f()* can easily be calculated only by additions. With our example *D2* = 2 added to *D1* = 7 gives *D1* = 9 and that sum added to the last function value *f(3)* = 8 gives the new value *f(4)* = 17 and so on. All additions may be done with mechanical adders that store the intermediate results and they should be done by machine to avoid errors, because a possible error runs through all calculations that follow. Of course the calculus of finite differences provides not only extrapolation starting from a given value as shown above, but also interpolation between two values and many more algorithms and possibilities that cannot be explained here.

A difference engine is adapted to this algorithm shown above, it is build of serial connected adders that store and transfer intermediate results. Such a serial machine should not be mixed up with a double and parallel working machine used especially in geodesy.

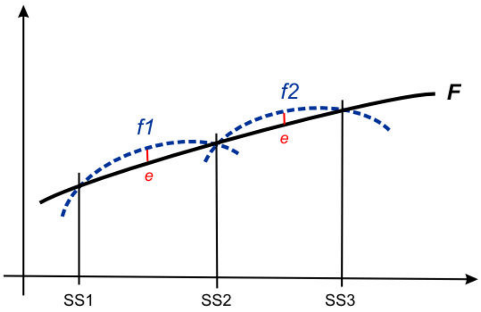


Figure 2 - Approximation of a function

Logarithmic, trigonometric and some other functions cannot be expressed by polynomial functions, but fortunately they can be approximated. To approximate the function *F* in fig. 2 we select so called supporting points SS on it, calculated with great accuracy or taken from other tables, and choose the coefficients of the polynomials *f1*, *f2*... thus that they lead through the selected points. Care has to be taken of the maximum errors *e*.

Since logarithmic and trigonometric functions can be approximated by polynomials, such a difference engine is more general than it appears at first. A historical remark: Gaspard de Prony used the methods of differences when he organized calculations for the *Tables du Cadastre* like producing goods in a factory at the end of the 18th century and later Babbage thought of them.

The first difference engine we meet in the 20th century is that of Christel Hamann in Berlin who at that time has been well known for his desk calculators *Gauss* and *Euklid*.

Christel Hamann

Shortly before 1900 the astronomers Julius Bauschinger and Jean Peters decided to calculate new logarithmic and trigonometrical tables with eight figures to meet the constant increasing requirements for greater accuracy in astronomy and geodesy. First discussions between Julius Bauschinger and the mathematician Heinrich Bruns took place in 1904. Bruns acted as a consultant for all problems related to calculation of the tables. Both decided not to recalculate again all values but to use the method of interpolation between known values with second differences. In the first years they thought of using a Burroughs adding machine. Later, in spring 1908 when the first calculations started in preparation for the mechanical interpolations, Hamann was asked to design and build a machine for the aimed purpose. Only one year later Hamann delivered his unnamed difference engine that surpassed all expectations. As far as we know the machine was used only for one complete run. The first table, derived from the results, was published in 1910 [1, 5:#197.0].

Next we will have a closer look to the construction of the machine.

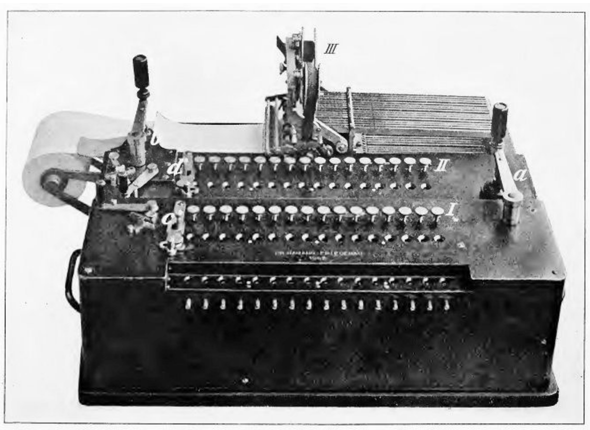
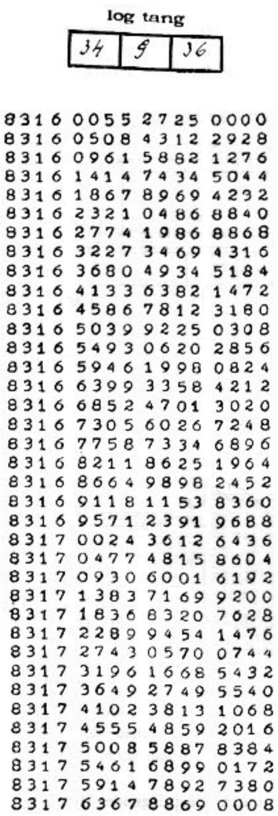


Figure 3 - Hamann's difference engine

The machine (fig. 3) must have been a large and heavy one, I reconstructed a weight of about 40 to 50 kilograms. It is divided into three parts: in the first adder placed next to the user the second difference is added to the first difference. With the second adder in the middle part this sum is added to the intermediate result and the third section, a printer, prints the result onto a strip of paper. All differences and the result can be set, operated and printed with sixteen places. Each of the two adders is driven by its own handle. With use of the printing device errors in copying from result register to first script are avoided.

The construction of the machine and the way the logarithms were calculated are described in the foreword of the first volume in the first German edition. In the English edition the description of the machine is missing.

All printed strips which the machine produced were given to Astronomisches Rechen-Institut in Berlin. The papers are lost, only a copy of a single stripe survived (fig. 4). It shows the interpolation between log tan 34° 9' 36'' and log tan 34° 10' 12''. The function values should be read as 9.831... – 10. The ten is omitted and the decimal fraction is regarded as an integer number.



8316	0055	2725	0000
8316	0508	4312	2928
8316	0961	5882	1276
8316	1414	7434	5044
8316	1867	8969	4232
8316	2321	0486	8840
8316	2774	1986	8868
8316	3227	3469	4316
8316	3680	4934	5184
8316	4133	6382	1472
8316	4586	7812	3180
8316	5039	9225	0308
8316	5493	0620	2856
8316	5946	1998	0824
8316	6399	3358	4212
8316	6852	4701	3020
8316	7305	6026	7248
8316	7758	7334	6896
8316	8211	8625	1964
8316	8664	9898	2452
8316	9118	1153	8360
8316	9571	2391	9688
8317	0024	3612	6436
8317	0477	4815	8604
8317	0930	6001	6192
8317	1383	7169	9200
8317	1836	8320	7628
8317	2289	9454	1476
8317	2743	0570	0744
8317	3196	1668	5432
8317	3649	2749	5540
8317	4102	3813	1068
8317	4555	4859	2016
8317	5008	5887	8384
8317	5461	6899	0172
8317	5914	7892	7380
8317	6367	8869	0008

Figure 4 - Copy of a printed stripe from Hamann's machine

The two main points log tan 34° 9' 36'' (first line) and log tan 34° 10' 12″ (last line) are either calculated with high precision or taken from Briggs-Gellibrand *Trigonometria Britannica* (1633 and later). Logarithms for numbers they took from Briggs' *Arithmetica Logarithmica* (1624 and later) and other works. If either the original value or the result of calculations with differences is incorrect the next supporting points will not meet close together.

In the preserved example the used starting point is
log tan 34°09'36'' = 9,8316 0055 2725 – 10
with the differences
d1 = 0,0000 0453 1587 2928
and
d2 = -0,0000 0000 0017 4580

It is reported that a trained human computer could do the input to the machine and the whole interpolation in five minutes.

Now we should do some own calculations: we interpolate nine values between log 169500 and log 169510. The values log 169500, log 169510 and log 169520 we take from a table with high precision. For the large intervals between 169500, 169510 and 169520 we get the differences

```
log 169500 = 5.2291 6970 2539
                D11 = 0.0000 2562 1338
log 169510 = 5.2291 9532 3877
                D2  =-0.0000 0000 1512
                D12 = 0.0000 2561 9826
log 169520 = 5.2292 2094 3703
```

For the nine values between 169500 and 169510 we have to use the differences
d2 = 0.01*D2 + ...   = -0.0000 0000 0015 1200
d1 = 0.1*D1 – 0.045*D2 + ... = 0.0000 0256 2201 8400

The red numbers are set to the machine as starting points. Next we start a virtual machine and the output looks like this

2291 7226 4740 8400 ≡ log 169501
2291 7482 6927 5600 ≡ log 169502
2291 7738 9099 1600
2291 7995 1255 6400
2291 8251 3397 0000
2291 8507 5523 2400
2291 8763 7634 3600
2291 9019 9730 3600
2291 9276 1811 2400
2291 9532 3877 0000 ≡ log 169510

and this is how the appropriate row looks like in the table, rounded to eight places:

N.	0	1	2	3	4	5	6	7	8	9	d.
16950	229 16970	17226	17483	17739	17995	18251	18508	18764	19020	19276	256
51	19532	19789	20045	20301	20557	20813	21070	21326	21582	21838	256
52	22094	22351	22607	22863	23119	23375	23631	23888	24144	24400	256
53	24656	24912	25169	25425	25681	25937	26193	26449	26706	26962	256

For every interval, new differences must be calculated and they differ from interval to interval. That is why in the first year when the engine was build, about four human computers did nothing else but calculate differences for the later use with the machine. Since the supporting points have been either recalculated with high precision or taken from Briggs' table or at least compared with Briggs we can say from another point of view Briggs' tables have been enlarged by Bauschinger and Peters, not replaced.

Ten years later Peters used the printed output of Hamann's machine again and produced ten figures tables [7, 5:#199.3]. Why he published the new tables is described best with the words of *Encyclopaedia Britannica*, edition 1911. Under the heading 'Mathematical Tables' the dictionary writes : '*A copy of Vlacq's Arithmetica logarithmica (1628 or 1631), with the errors in numbers, logarithms, and differences corrected, is still the best table for a calculator who has to perform work requiring ten-figure logarithms of numbers, but the book is not easy to procure, and Vega's Thesaurus has the advantage of having log sines, &c., in the same volume.*'

Hamann's machine is regarded to be lost since the Twenties of last century, even the construction drawings could not be found. Only a picture of the machine, shown in fig. 3, the only one we have, and the above shown copy of a small printed sheet of paper survived.

Logarithms with eight places in 1910, with ten places in 1920 – no effort seemed to be sufficient. The next, the last and highest step in table making during the 20th century followed soon.

Alexander John Thompson

During the Twenties of the last century Alexander John Thompson, at that time member of staff in the General Register Office in London, decided to calculate a twenty figures logarithmic table [9]. Between 1924 und 1952, with a longer break during World War II, parts of the table appeared as nine booklets in Pearson's series *Tracts for Computers* [first part 5:#199.4]. In 1911 Karl Pearson (1857 – 1936) founded the world's first University Statistics Department at University College London and established the discipline of mathematical statistics. Origin and purpose of this table are described best with the words of the publisher Cambridge University Press in a summary of the book: *'This work of Dr Thompson's is an attempt to commemorate in a worthy manner the first great table of common logarithms, which was computed by Henry Briggs and published in London in 1624. It brings together the series of nine separate parts, issued between 1924 and 1952 from University College, London, in Karl Pearson's Tracts for Computers series. The main table, which consists of the common logarithms to twenty decimals, of numbers up to 100,000, is accompanied by differences of even order. It is likely to be used chiefly in the computation of other mathematical tables, and will facilitate the work of the large calculating machines now being developed. For these purposes values of 15 to 20 figures are often required. The table is preceded by a very full introduction which describes methods of interpolation and the mode of construction, and provides some useful auxiliary tables.'*

The intended commemoration is explicitly expressed in the addendum to the whole title …Issued by the Department of Statistics, University College, London, to commemorate the Tercentenary of Henry Briggs' publication of the *Arithmetica Logarithmica*, 1624.

Thompson not only commemorates Briggs' first big table three hundred years ago, he also lifts himself to the level of Briggs. When the reader opens the book he finds two title pages side by side. On the left side Briggs' old title page from 1624 is placed and on the right side one sees Thompson's title page in modern letters. Furthermore for his work Thompson composed the Latin name *Logarithmetica Britannica*, derived from the titles of Briggs' tables *Arithmetica Logarithmica* and *Trigonometria Britannica* with the adjective logarithmic here changed to a noun. With respect to his extensive work – twenty figures logarithms for the numbers 10.000 (1) 100.000 – in my opinion he had some rights to do so.

When he started neither new industrial manufactured nor historic difference engines weren't available, so after having worked for a short time with a single calculating machine of type Odhner he decided to build one by himself. Assistance he got from the agent for Triumphator calculating machines in England, whom he thanked for his help in the introduction to his work. This expression of thanks is the only source for us to know that he used Triumphator calculating machines, produced by Triumphator Rechenmaschinenfabrik GmbH in Leipzig, Germany. The only known picture of the composed machine is shown in the introduction to the table (fig. 5).

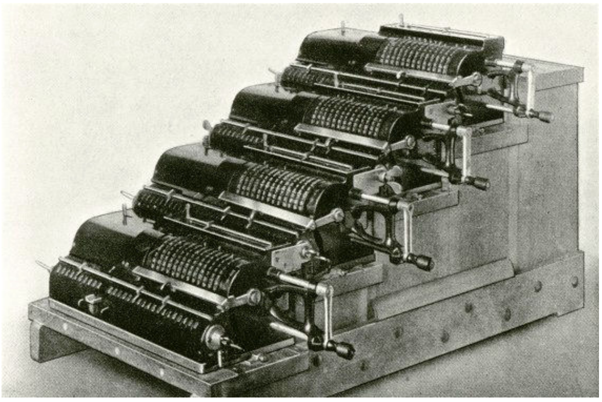


Figure 5 - Thompson's difference engine

Four single machines are arranged on a stepped wooden base. The fifth highest step may lead to the assumption that he thought of a fifth machine. Results are transferred downstairs from a result register to the upper part of the input register of the next lower machine. That is why the single machines are arranged one behind the other and in increasing height. Thompson didn't call his machine a difference engine, he named it with the oppositional expression *integrating and differencing machine* with respect to the fact that the original meaning of to integrate is to sum up and that the machine is used for calculation with differences. In Thompson's opinion his machine is the only one with its peculiar design – and he is right to think so – and therefore he doesn't intend to explain all technical details and procedures how to work with it. If however we follow his explanations more details not mentioned by him can be reconstructed.

If we denote with D1..D4 the differences and their orders and with M4 the highest, with M3 the next lower machine and so on, a calculation with four differences runs as follows:

D4 is set in the input device of M4,
D3 is set in the result register of M4,
D2 is set in the result register of M3,
D1 is set in the result register of M2,
the last function value is set in the result register of M1.

Next we

add D4 to D3 in M4 and transfer D3 to input device of M3,
add D3 to D2 in M3 and transfer D2 to the input device of M2,
add D2 to D1 in M2 and transfer D1 to the input device of M1,
and finally
add D1 in M1 to the last function value.

When Thompson had finished his work in the following decades nobody knew what had happened with his machine. In 2007 I had luck finding Thompson's machine in the cellar of a Statistical Department in London and I asked a friend of mine. a photographer in London, to take some pictures. The machine has retired but is still alive and sometimes is used to demonstrate to the students how logarithms were calculated by grandfather in former times.

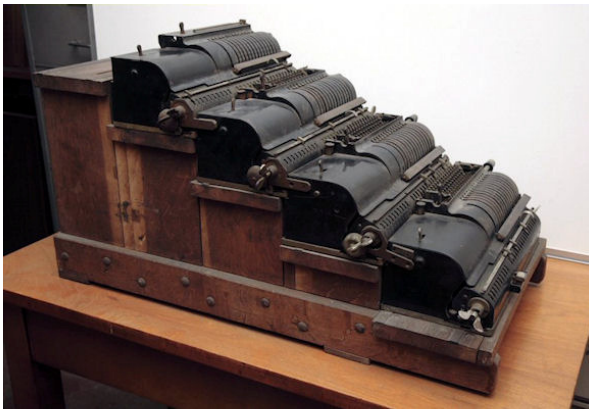


Figure 6.1 - Thompson's difference engine, left side

The photographs I got allow a closer look to the machine.

Fig. 6.1 displays the left side of the machine, fig. 6.2 the transfer unit from the result device of the upper machine to the input device in the lower machine. The result units are locked with a plate, otherwise the transfer mechanism wouldn't work.

Fig. 6.3 shows an input device with thirteen levers. The lettering *Triumphator* on the left side is cut by the zeroing mechanism. This detail is a second indication that they used Triumphator machines with a nine places input and enlarged them to thirteen places. The result device holds eighteen places, but with a fixed result device it only can be used up to thirteen places plus a possible carry.

Figure 6.2 - Thompson's difference engine, the transfer mechanism

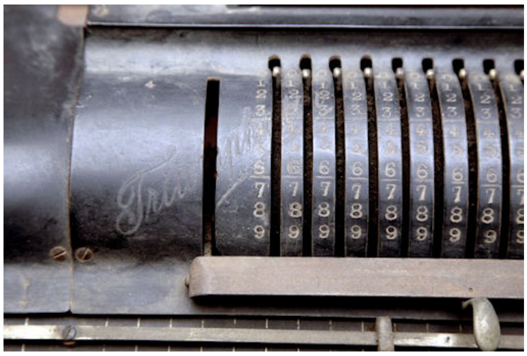


Figure 6.3 - Thompson's difference engine, the input device

To calculate logarithms with twenty figures one needs at least twenty-three or twenty-four figures to avoid errors in rounding. A question arises: how to calculate twenty-three figures with a machine with thirteen input levers? The answer astonishes: Thompson calculated twice, an example will demonstrate how he proceeded.

Assuming based on known values and a difference he has to calculate a new logarithm like in
log(N+1) = –log(N–1) + 2log(N) + d2log(N)
for N = 15455 with
A = log(N–1) = 18904 09079 09009 92819 (actually 4,18904...)
B = log(N) = 18906 90093 99323 73840
C = d2log(N) = –00000 00018 18219 42567

In the first run he works with the ten figures on the right side of all summands, in the second run he processes the left figures in the same summands included the carry from the first run. The following record summarized the whole process.

		2^{nd} run R2	1^{st} run R1
A	–	1890409079	– 0900992819
B	+	1890690093	+ 9932373840
B	+	1890690093	+ 9932373840
C	–	0000000018	– 1821942567
sum		**1890971090**	**17141812294**

We get log15456 = 18909 71090 71418 12294 (actually 4,189097...)

In this example I use twenty figure numbers because I don't know the values with twenty-three or twenty-four figures he really used.

Actually not all logarithms were calculated with differences, abbreviations can save labour. So if log 80.000 to log 100.000 are already known, log 40.000 to log 50.000 may simply derived with the expression log N = log 2N – log 2. On the other side very much effort was necessary to control and minimize possible errors. For that purpose Thompson calculated and used differences up to 10th order.

In a short time it is impossible to explain here all algorithms, formulas or abbreviations Thompson used to calculate differences and logarithm – his original description includes 54 large folio pages – but the two used examples will be enough to demonstrate what a difficult and hard task Thompson undertook. He even bought a monotype keyboard, typed the final results on punched tape for auto-print and compared the results by himself.

N, 15400—15500

N	log N	δ^2 —	δ^4 —
15450	18892 84837 60853 44725+	18 19396 45715+	4573
51	95 65925 26398 56608	9160 95944	72
52	18898 46994 72782 72546	8925 50744	71
53	18901 28046 00241 37741	8690 10116	70
54	04 09079 09009 92819	8454 74057	68
55	06 90093 99323 73840	8219 42567	67
56	09 71090 71418 12294	7984 15644	66
57	12 52069 25528 35105−	7748 93287	65−
58	15 33029 61889 64628	7513 75495−	64
59	18 13971 80737 18657	7278 62267	63

Figure 7 - Small part of Logarithmetica Britannica

Figure 7 shows a small section of *Logarithmetica* with the columns numbers N, logarithms logN, and second and forth differences d^2 and d^4.

Since Thompson based his calculations on Briggs' *Arithmetica*, his work gives an extensive table of errors found there. Another source he used for supporting points is Sharp, 1717 [8, 5:#65.0], who gives logarithms of numbers 1 to 99 with sixty-three figures and of prime numbers 101 to 1097.

Thompson's *Logarithmetica* is an extensive work, placed on the level of Briggs and, when it was finished, some years later followed by desk and pocket calculators, not to speak of electronic calculators, which made it finally useless.

His difference engine remained to be the last machine built for the specific methods of calculations with differences. At the end of the Twenties the commercial market offered machines for more general purposes and that could be used as difference engines too. In relation to commercial machines Leslie John Comrie (1839 – 1950) should be mentioned.

Leslie John Comrie

Being an astronomer and mathematician and member of the Nautical Almanac Office he soon became an expert in all aspects of calculating tables and therefore tested various types of machines [4.1 – 4.5 in a selection]. He neither invented nor built a difference engine, but he investigasted and demonstrated how to use commercial machines for the special purposes of table making.

At the same time when Thompson started his work and built his own machine, Comrie published an article on how to use Brunsviga Dupla for calculations with second differences [4.1]. The Dupla, manufactured by Brunsviga-Maschinenwerke in Braunschweig, Germany, between 1927 and 1930, is not a difference engine, it is a single calculating machine, but with peculiar properties. With it you can add a number from input device to one of the two result registers and transfer the content of both result registers back to the input levers.

Some years later he used a Hollerith (later IBM) accounting equipment for tabulating. For a National accounting machine and a Burroughs Class 2 machine that followed he developed algorithms for calculations with differences too [4.4].

It was Comrie's valuable contribution to show that cheap commercial accounting machines could be used as difference engines and thus he revolutionised the art of table making until the new technology modern computers came into use.

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Sources

Fig. 3: [1] foreword
Fig. 5 & 7: [9]
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