A reconstruction of Felkel's tables of primes and factors (1776)

Introduction

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(...) certainly the most remarkable-looking table we have seen.
J. W. L. Glaisher [38, p. 36]

[Felkel's] table, both on account of its rarity and its confused arrangement, is practically useless.
James Glaisher [37, p. 25]

1 Lambert's tables project

In 1770, Johann Heinrich Lambert (1728–1777) published a collection of tables containing in particular a table giving the smallest factor of all integers up to 102000 and not divisible by 2, 3, or 5 [54].

In that table, Lambert wrote that "it would in fact be desirable if we could obtain the factors of a number from 1 to 1 000 000 and even beyond, merely by opening a table." [1] This note had some tangible effects, first in that a number of persons sent to Lambert various tables they had made before, and second in that others started to extend Lambert's table, often seeking some fame. Felkel was one of those.

2 Anton Felkel (1740–ca. 1800?)

Anton Felkel[2] was born in 1740 in Kamenz (Kamieniec Ząbkowicki in Poland). At the age of seven, he went to the piarist school in Weißwasser (Bílá Voda in Czech Republic) and this is where he learned German and Latin. He decided to become a priest and in 1765 he therefore went to the monastery of Sagan (Żagań in Poland). But soon afterwards, he gave up this idea and decided to travel. He first went to Rome, and then to Vienna. He studied law, then became teacher in 1767 at the *St. Stephan Bürgerschule* in Vienna. He then returned to Sagan in order to learn about Felbiger's[3] school methods. When he came back to Vienna, he applied these methods in the orphan school headed by Ignaz Parhamer (1715–1786) [25]. In 1771, he became teacher in the newly opened *Normalschule* in Vienna, where he was teaching at least until 1776 [71, pp. 21–34].

According to Luca's contemporary biographical notice, Felkel's first works were tables, printed after 1771. It was at the beginning of the 1770s that Felkel studied mathematics with Lukas Ebe, a teacher in Linz, and Wilhelm Bauer, who was professor in Vienna.[4] This led him, in 1775–1777 to work on the tables described here.

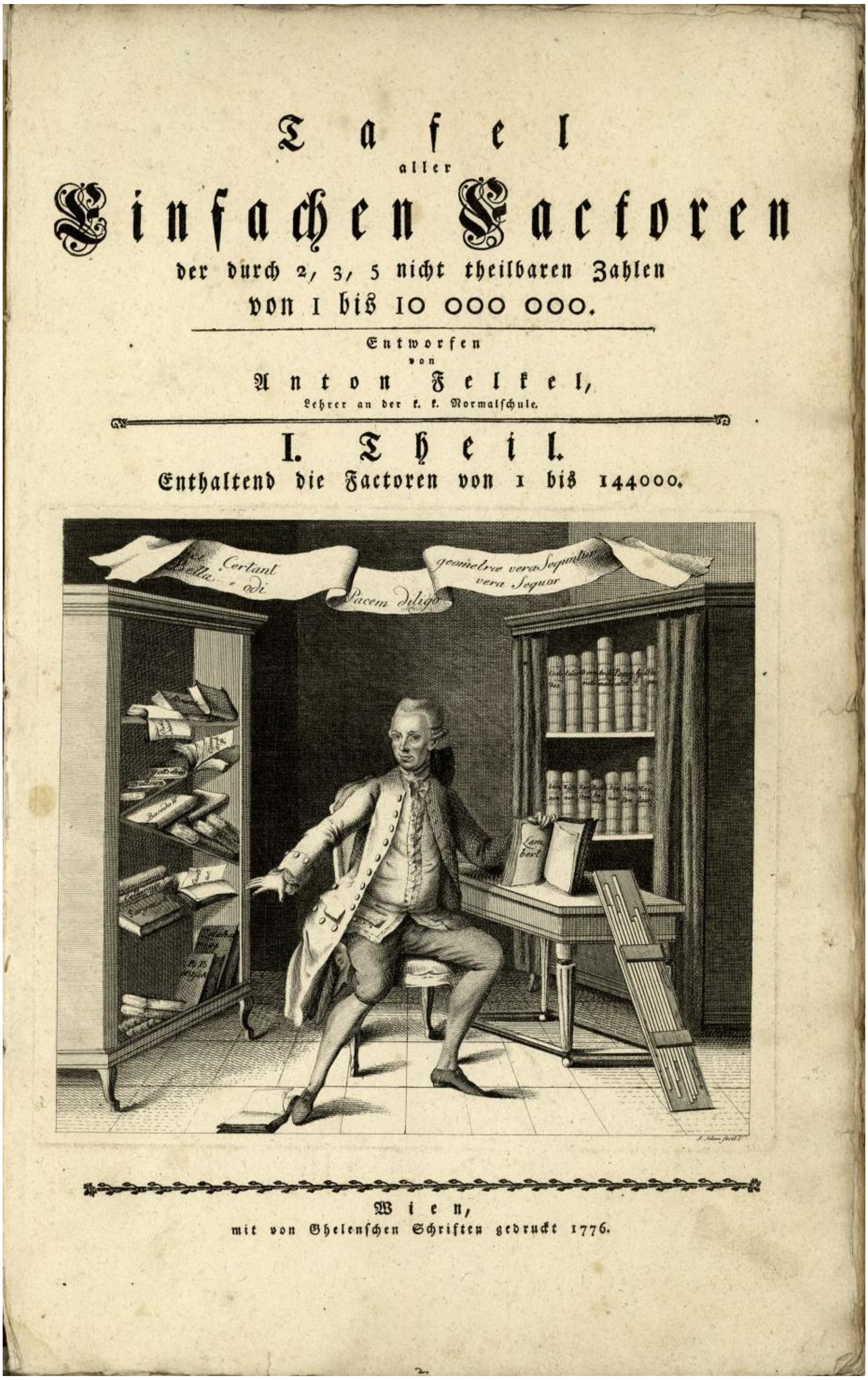
Felkel also wrote a book on the theory of parallels in 1781 [32], perhaps inspired by Lambert.

[1] *Es wäre in der That erwünscht, wenn wir von 1 bis auf 1000000 und noch weiter die Theiler der Zahlen durch blosses Aufschlagen einer Tafel haben könnten [54, p. 9].*

[2] The main sources on Felkel are the biographical notice by Luca [25], Felkel's letters in Lambert's correspondence [14, 15], the introduction to the translation of Lambert's tables [56], and the school records.

[3] Johann Ignaz von Felbiger (1724–1788) was abbot of the Augustine monastery in Sagan.

[4] On Ebe (1753–181?), see [73], and on Bauer (1742–1825) see [71, pp. 55–60]. Both Ebe and Bauer published textbooks of mathematics.



Tafel
aller
Einfachen Factoren
der durch 2, 3, 5 nicht theilbaren Zahlen
von 1 bis 10 000 000.

Entworfen
von
Anton Felkel,
Lehrer an der k. k. Normalschule.

I. Theil.
Enthaltend die Factoren von 1 bis 144000.

Wien,
mit von Ghelenschen Schriften gedruckt 1776.

Figure 1: The engraving on the first page of Felkel's table to 144000. This presumably depicts Felkel [15, pp. 222–223], [9, p. 337]. (source: Göttinger Digitalisierungszentrum)

In 1784, he resumed his work on factor tables after having laid it aside for eight years. In 1787, he was director of the *gräfl. Thun'schen Armenschulen* in Bohemia, a charity school [15, pp. v–vi].

In 1791, he went to Lisbon [56, preface, p. VII], [41, pp. 306–307]. There he became the head of the Casa Pia charity school.[5] During that period, he continued to work on tables and was in touch with the Portuguese mathematician Garção Stockler (1759–1829). Stockler wrote him a letter about the computation of factors and it sheds some light on Felkel's research at that time [36, 6].[6]

According to Cantor, Felkel invented several machines, in particular a reading and speaking machine [20]. It is presumed that Felkel died in Lisbon around 1800.

3 Felkel's first tables

Felkel's projects evolved quickly between 1774 and 1777. Felkel writes that he started to read Euclid in 1774 [15, p. 489]. In October 1775, he started to read Euler's algebra [15, p. 491]. Felkel started his work on table of factors in November 1775 [15, p. 156] and eventually, especially following the race with Hindenburg, his aim was to give a table of factors to 10 millions. Felkel recounted this period as that of the "factor disease"[7] [15, p. 489].

Felkel set out to give all simple factors of all integers not divisible by 2, 3, or 5. In order to achieve this goal, he had to save as much space as possible, and he naturally decided to use symbols for various primes. Felkel also decided to publish his table by subscription.

Felkel wrote that the first table was computed to 2 millions from scratch in 18 months,[8] but war troubles, deaths and other problems prevented the printing of going beyond 408000, and that at the same time it had to become destroyed [15, pp. 123–124], [39, p. 118].[9]

Felkel's table was published in three parts.[10] The first part to 144000 was first pub-

[5] The Casa Pia school may have archives on Felkel, and this should be investigated.

[6] The letter is undated, but was probably written around 1795. In 1799, Felkel also wrote to Joseph Banks, the president of the Royal Society in London, about his studies of factors [24, p. 325], and the content of the letter is possibly related.

[7] *Faktorenkrankheit*

[8] In 1798, Felkel wrote of 16 months [56, preface, p. VII].

[9] Palamà writes that the manuscript to 2 millions passed into the possession of the Imperial Treasury in Vienna [61]. But Zach, who is cited by Palamà, writes that Felkel obtained the manuscript back from the war office [75]. Gauss also mentions Felkel's work in the preface to Dase's seventh million and writes that Felkel had his table ready in manuscript up to 2 millions [23].

[10] Archibald knew of complete copies of Felkel's table in the Graves library in London and in the library of the University of Tübingen. Moreover, the first two parts are also at Göttingen and the first part is at the Royal Society in London [9]. Archibald also mentioned a photographic copy of the Tübingen copy at Brown University. Digitized versions of the tables to 336000 are available on the web. The only actual copy that we saw is located in the library of the observatory of the University of Vienna, and it does extend to 408000. The volumes have the approximate dimensions 28.7 cm × 42 cm. In this copy the second and third parts have been bound together, with no intermediate title for the third part. The second volume bears the same title as the second copy in Goettingen, namely stating only that the range

lished in German in 1776, then in Latin in 1777. The two other parts (from 144001 to 336000 and from 336001 to 408000) were published in Latin, probably in 1777. Felkel stopped working on the tables around April 1777, and he testified himself that he worked for 16 or 18 months on that topic. Since very few copies of Felkel's table were sold, the whole printing, with only a few exceptions, had to be destroyed, although the exact circumstances are somewhat obscure. In 1784, Felkel explained that because of war troubles, deaths, and other fates, the copies had to suffer destruction [15, pp. 121–124]. It was Zach who wrote in 1800 that the printing was converted to "Patronen-Papier" for the last war with the Turks [75]. But there should be doubts about this assertion, as the Austro-Turkish war took place from 1787 to 1792. It is therefore not totally clear which war troubles Felkel had in mind in 1784, but they were not the war with the Turks.

Felkel's layout appears close to the one proposed by Euler in 1775 [26], but he was not influenced by it, as Euler's article was probably not yet published when Felkel started his work [15, pp. 488–489]. Moreover, Felkel himself wrote that by 1785 he had still not yet seen Euler's article [15, p. 501].

Felkel then left the tables of factor aside for eight years. He only resumed working on factor tables in 1784, chiefly as a consequence of the edition of Lambert's correspondence (see § 6.1).

3.1 The coding of the primes

Felkel only published his table to 408000. In our reconstruction, we have extrapolated Felkel's scheme and named all primes up to 10 millions (and a little more) and used these names in the decompositions. Felkel's table is difficult to use, as it requires to look up primes either in the main table itself, or in an auxiliary table which is quite long. Felkel's auxiliary table of primes was only one page long for the interval 1–144000, but Felkel made use of primes which are not in this auxiliary table. The complete auxiliary table for the primes to 10 millions is 289 pages long.

In Felkel's published tables to 144000, the only primes which appear with a multiplicity greater than 1 are those of the first column of the auxiliary table (figure 2), namely primes up to 523. But in the table to 408000, there are cases where a greater prime is squared. These primes have names made of more than one symbol. For instance, $292681 = 541^2$ and 541 is encoded with αa. The question arose whether Felkel wrote $(\alpha a)^2$ or αa^2. The inspection of his table reveals that he merely adds an exponent, but does not use parentheses. This may seem strange, but it is in fact not ambiguous, because Felkel on the one hand separates consecutive factors with a small space, and on the other hand that $\alpha \cdot a^2$ is not a valid decomposition, because α can not stand for an isolated prime.

The primes until 523 are indicated by a unique symbol. Starting with 541, prefixes are used. The first series of prefixes runs the lowercase Greek alphabet from α to ω, and so we have $\alpha a = 541$, $\alpha b = 547$, etc. The next series runs the uppercase Greek alphabet ($Aa = 22307, \ldots, Ba = 23321$, etc.). Then follows the lowercase Greek alphabet prefixed by a parenthesis ("$(\alpha a$"$= 47527$, etc.), the uppercase Greek alphabet prefixed

is 144001–336000. The copy at the Graves library in London is probably similar, except that the first part is in Latin.

by a parenthesis, the Greek alphabets postfixed by a parenthesis, the Greek alphabets prefixed by a bracket, the Greek alphabets postfixed by a bracket, the Greek alphabets between a pair of parentheses, the Greek alphabets between a pair of brackets, and the Greek alphabets prefixed by a half-bracket ⌈. We assume that Felkel would have continued with the Greek alphabets postfixed by a half-bracket ⌉, but what would have come next is not known. If a group of prefixes $\ldots\alpha$-$\ldots\Omega$ is called an alphabet, he would have needed $\lceil\frac{23\times289-49}{48}\rceil = 138$ alphabets, which are difficult to envision with only one common symbol.[11] Felkel may have planned to use the brackets and parentheses (,), [,], {, }, ⌈, ⌉, ⌊, ⌋, ⟦, ⟧ and perhaps other special characters. With a pair of 12 such symbols, 144 combinations are possible, which would be sufficient to name the 138 alphabets, but these symbols would have had a drawback, in that primes often need to be located, and locating them requires locating the alphabet in a sequence. This is much easier if the prefixes follow a regular pattern. Therefore, although we could have used such combinations of patterns in our reconstruction, we have actually started to use a simple 3-symbol prefix from 523049 on. This prefix is made of a digit followed by a pair of Greek letters. The first two symbols define the alphabet and we can have $10 \times 48 = 480$ alphabets, of which we only use 130, not counting the initial alphabets used by Felkel. Our extended prefixes are $0\alpha\alpha$, $0\alpha\beta$, $\ldots$, $0\alpha\omega$, $0\alpha A$, $\ldots$, $0\alpha\Omega$, $0\beta\alpha$, $0\beta\beta$, $\ldots$, $0\beta\omega$, $0\beta A$, $\ldots$, $0\beta\Omega$, $\ldots$, $0\Omega\Omega$, $1\alpha\alpha$, $\ldots$ Our last but one prefix is $2\mathrm{K}\chi$ starting with 10 000 439, and our last prefix is $2\mathrm{K}\psi$

The first prime with a 3-symbols name is $(\alpha a = 47527$ and it nevers occurs with a multiplicity greater than 1. Only primes smaller or equal to $3137 = \delta\mathfrak{w}$ can occur with a multiplicity greater than 1 between 1 and 10 002 000.

We should observe that Felkel's interest was also in counting the primes, and Felkel's naming scheme makes it relatively easy. In order to find the position of a prime, one merely needs to know the position of the alphabet in the sequence of alphabets and the position of the prime within the alphabet.

In the fifth volume of Lambert's correspondence, Bernoulli appears critical of the use of letters by Felkel, and concludes that the demand for a different arrangement is warranted [15, pp. 230–231].

Several other tables made use of symbols for numbers. This was in particular the case of Jakob Philipp Kulik, who may have been inspired by Felkel, at least indirectly [51]. However, for each composite number, Kulik only gives the smallest factor, whereas Felkel gives the whole decomposition. Moreover, Felkel gives names to all the primes, and Kulik only named those that he needed, and the greatest prime that could occur in his tables was 10009.

3.2 The table of primes

In his table giving the factors for numbers up to 144000, Felkel gave on one page the prime numbers from 1 to 20353 (figure 2). This table names all the primes which are

[11]Nevertheless, an alternative would be the use of the number notation developed in the late 13th century by Cistercian monks and described by David King [47]. With this notation, we could express a number up to 9999 by a unique symbol.

factors in the table, but it does not show Felkel's scheme beyond 20353. Of course, this table is not even enough for naming all factors up to 10 millions and beyond.

In our reconstruction, we have chosen to extend Felkel's table of primes so as to cover all prime numbers up to 10 001 921. Using 100 lines and 23 columns, this list covers exactly 289 pages. There is only one prime number which appears in the complete table to 10 002 000 and not in the first table, namely 10 001 963, which is named 2Kψa in our naming scheme (figure 3). It didn't seem necessary to add a whole page of primes for that sole case.

3.3 The basic layout of a factor page

When Felkel published the first part of his table, he also gave a one-page table showing the layout of his main table and useful to locate a given number (figure 4).

An examination of Felkel's table gives a feeling of irregularity in that the content of a cell can have three different forms:

- if a number is composed, all the symbols for all factors are given in sequence, from the smallest to the largest, and separated by a small space; these entries are justified to the right;
- the second form is made of a number followed by a dash and a letter, roman or gothic, lowercase or uppercase; this refers to a prime and the complete symbol for the prime is made of the current prefix (in the margin) and the suffix; these entries are justified to the left;
- the third form is a special case of the second form, for those primes using a prefix for the first time; the number for the prime is then followed by a dot, followed by the prefix, followed by the letter; these entries are also justified to the left.

In 1873, Glaisher gave a slightly incorrect description of Felkel's table. He wrote that Felkel only gave the last three digits of a composed number when there was enough space for it [38, p. 36]. In fact, Felkel only gives the last three digits on the first page (for all numbers) and for the primes on the subsequent pages. This is of course redundant, as the "coordinates" would be sufficient to identify the numbers.

3.4 The 1776 factor table and its completion

We have reconstructed Felkel's projected table of factors up to 10 002 000, but it has only been made with certainty from 1 to 408000. Afterwards, it is not totally clear what Felkel's conventions would have been for the prime alphabets. Towards the end of his table, Felkel used "⌈" as a prefix in an alphabet, and he would then presumably have used "⌉" as a postfix, but this is only a guess.

There are also anomalies in the page numbering. In the first part of the table (1–144000), Felkel started with 1 and numbered each page separately. In the second part, he

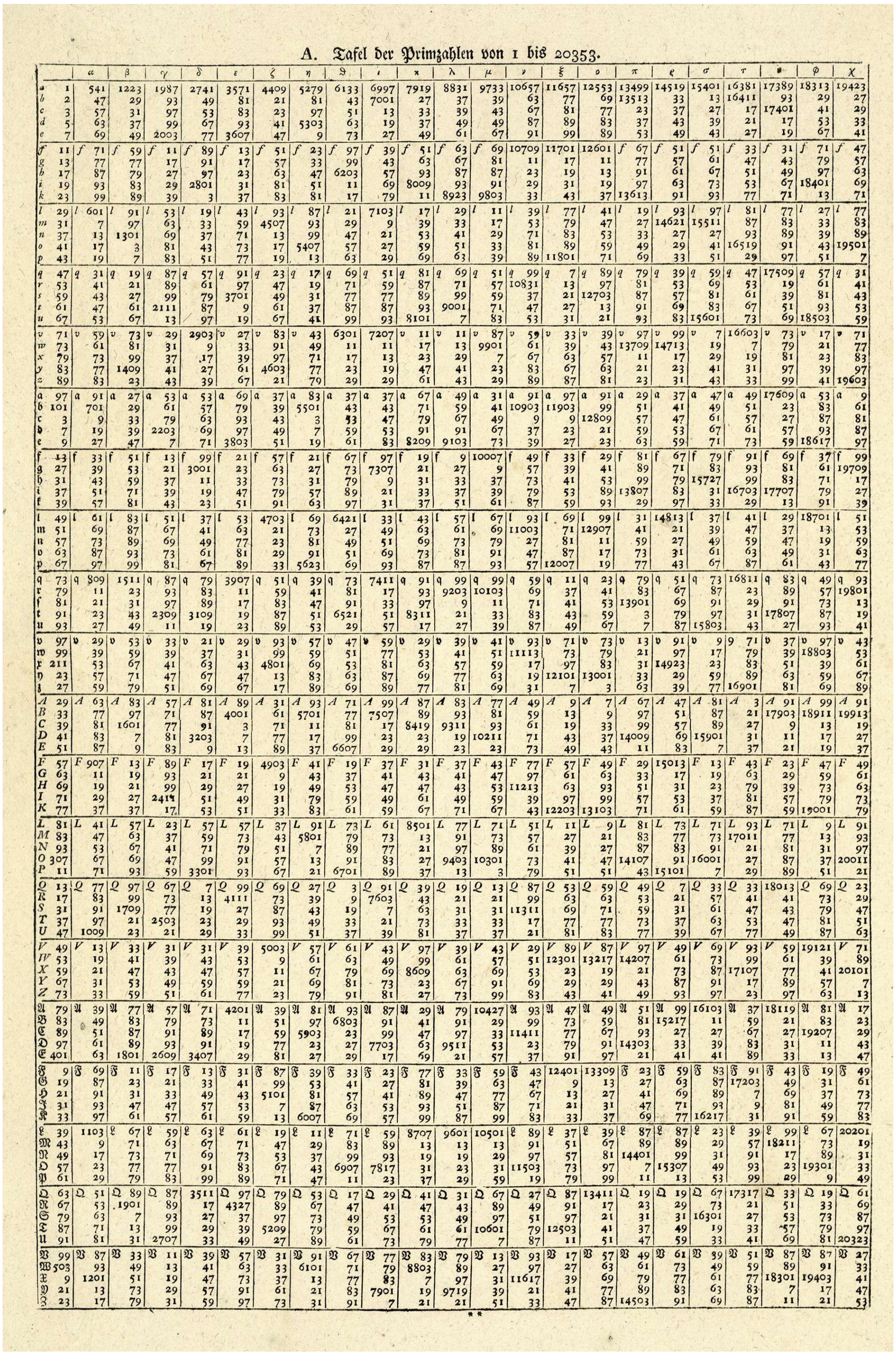


Figure 2: Felkel's table of primes from 1 to 20353. (source: Göttinger Digitalisierungszentrum)

89	$g\ 0\delta\Pi\mathfrak{R}$	$q\ [\mathrm{X}\mathfrak{e}$	$f\ 0\zeta\Psi P$	$\delta\mathfrak{t}\ \delta F$	$i\ w\ \iota w$	659-$\mathfrak{E}$	$e\ o\ \mathrm{N}w$	$F\ \Pi\mathfrak{Z}$	
90	$p\ \xi]\mathfrak{u}$	$e\ 0\xi\Lambda q$	$\alpha O\ \mu V$	$r\ [\mathrm{B}H$	687-$\mathfrak{F}$	$H\ \mathrm{O}\mathfrak{p}$	$g\ u\ \nu\mathfrak{K}$	$v\ \mathrm{K})\mathfrak{D}$	
91	701-$\mathfrak{G}$	707-$\mathfrak{H}$	$k\ \lceil\mathrm{E}B$	713-$\mathfrak{J}$	$f\ 0\zeta\Psi Q$	$e\ g\ \mathfrak{h}\ \alpha\mathfrak{w}$	$l\ [\eta]V$	$h\ n\ \sigma D$	
92	$z\ \mathfrak{e}\ \alpha Y$	$\mathfrak{L}\ \mathrm{A}\mathfrak{w}$	$q\ D\ \alpha D$	$t\ \mathfrak{f}\ \beta\mathfrak{f}$	$e\ m\ \Psi T$	$y\ \rho)\mathfrak{q}$	$O\ \mathrm{K}\mathfrak{X}$	759-$\mathfrak{K}$	
93	$e\ f\ \mathrm{A})B$	$\mathfrak{a}^2\ \alpha\mathfrak{E}$	$g\ i\ \Sigma\mathfrak{v}$	$\mathfrak{y}\ \mathrm{X}M$	777-$\mathfrak{L}$	779-$\mathfrak{M}$	$f\ 0\zeta\Psi R$	$e\ r\ \mathrm{E}I$	
94	791-$\mathfrak{R}$	$g\ h\ \mathfrak{p}\ I$	801-$\mathfrak{O}$	$e\ k^2\ n\ w$	807-$\mathfrak{P}$	$i\ m\ \tau I$	813-$\mathfrak{Q}$	819-$\mathfrak{R}$	
95	821-$\mathfrak{S}$	$f\ o\ u\ S$	$e^2\ h\ \xi\mathfrak{p}$	833-$\mathfrak{T}$	837-$\mathfrak{U}$	$l\ \mathfrak{u}\ \beta\mathfrak{C}$	$p\ \mathfrak{o}\ \beta\mathfrak{a}$	$f\ g\ k\ \delta\mathfrak{m}$	
96	$Y\ \mathrm{E}\mathfrak{J}$	$s\ [\kappa F$	861-$\mathfrak{V}$	$\alpha\mathfrak{c}\ \pi O$	$\mathfrak{K}\ \mathrm{A}\mathfrak{F}$	$\alpha G\ \nu\mathfrak{i}$	$e\ 0\xi\Lambda r$	$\mathfrak{s}\ (\vartheta q$	
97	881-$\mathfrak{W}$	$e\ \mathfrak{v}\ \iota\mathfrak{d}$	891-$\mathfrak{X}$	$f\ \alpha h\ \beta\mathfrak{u}$	$l\ [\eta]W$	$h\ 0\beta\beta\mathfrak{m}$	903-$\mathfrak{Y}$	$o\ z\ \delta a$	
98	$C\ \mathrm{TZ}$	$\beta\mathfrak{P}\ \eta f$	921-$\mathfrak{Z}$	$i\ \mathfrak{l}\ \delta\mathfrak{U}$	$g^2\ (\lambda\mathfrak{K}$	$e^2\ p\ q\ \mathfrak{b}$	$h\ m\ \varphi I$	$\mathfrak{w}\ (\gamma I$	
99	$k\ \lceil\mathrm{E}C$	$\delta\mathfrak{i}\ \delta R$	$n\ \mathrm{Y}]W$	$g\ x\ \mu b$	$e\ 0\xi\Lambda s$	$f\ \mathfrak{i}\ \vartheta G$	963.2Kψa	$\mathfrak{f}\ (\mathrm{N}\mathfrak{D}$	2Kψ
100	$e\ 0\xi\Lambda t$	$\beta q\ \iota O$	$f^2\ \mathfrak{h}\ \alpha q$	$v\ \mathfrak{r}\ \alpha\mathfrak{o}$	$k\ t\ \iota p$	$\beta t\ \iota\mathfrak{m}$	$\alpha C\ \nu X$	$e\ i\ \mathfrak{n}\ \mathfrak{S}$	

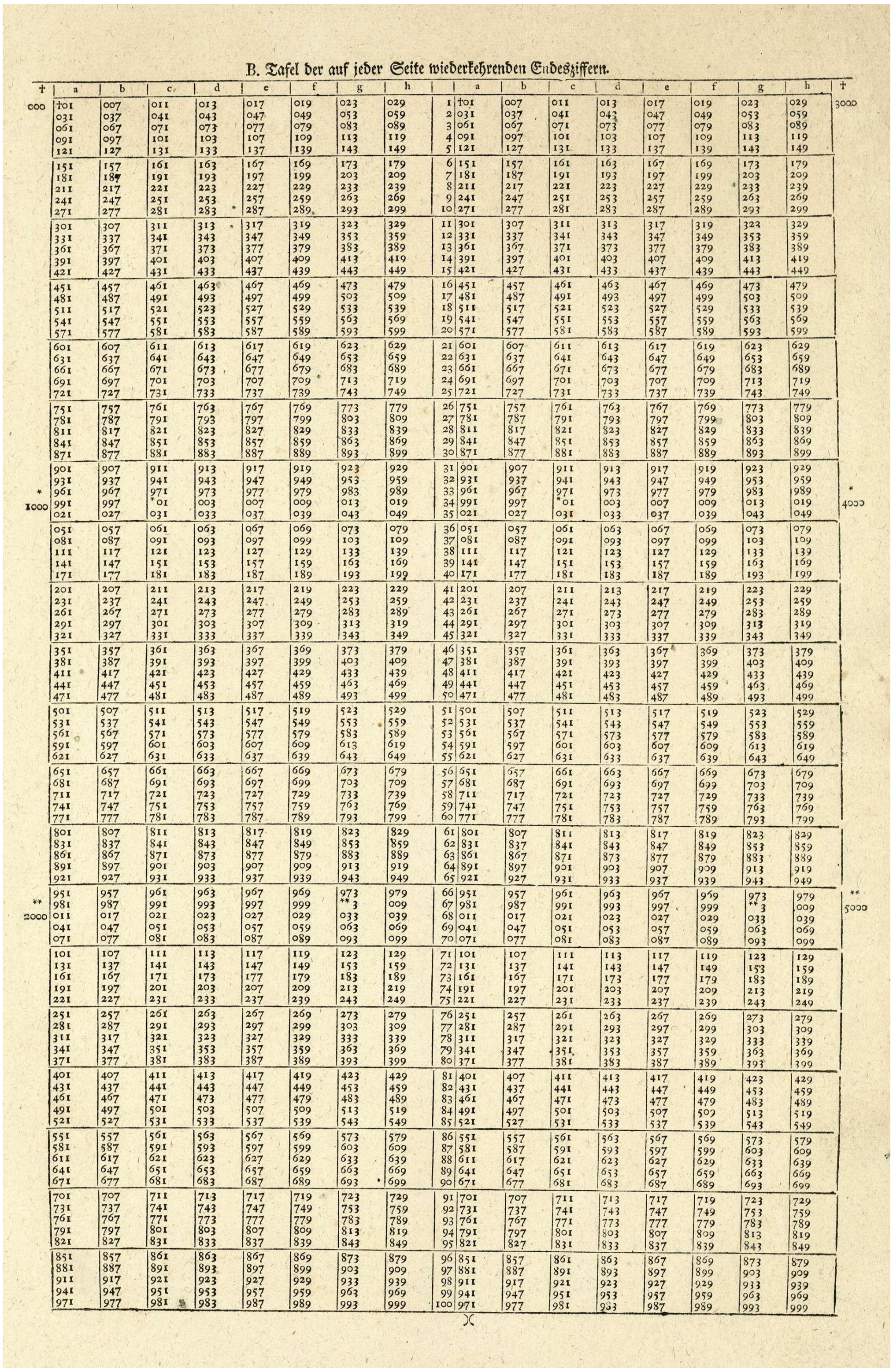
Figure 3: The end of the reconstructed table. This excerpt illustrates the three forms of an entry. Most of the entries are of the first form. For instance, the entry at the upper left gives a decomposition in two factors g and $0\delta\Pi\mathfrak{R}$. Two lines below, 701-$\mathfrak{G}$ represents a prime. The only occurrence of the third form is on the last but one line, with $963.2K\psi a$. Note that we have compressed the entry to make it fit the cell.

started again at 1, but numbered each column. We have reproduced this style faithfully, and extended it until the end of the table.[12]

Felkel has also changed the header of the pages. In the first part, the headers were in German (Factoren von ... bis ...). After 144000, the headers are in Latin (Factores ab ... usque ...). This too has been reproduced.

Felkel's table is certainly one of the best illustrations of the usefulness of a reconstruction. Checking Felkel's table without reconstructing it is extremely difficult, and spotting errors with our reconstruction has been much easier, although we have not gone into all details.

[12] Bernoulli remarked that if Felkel's first table is the first part of a table to 10 millions, then there would have been 70 such parts ($10\,000\,000/144000 = 69.44$) [15, pp. 222–223].

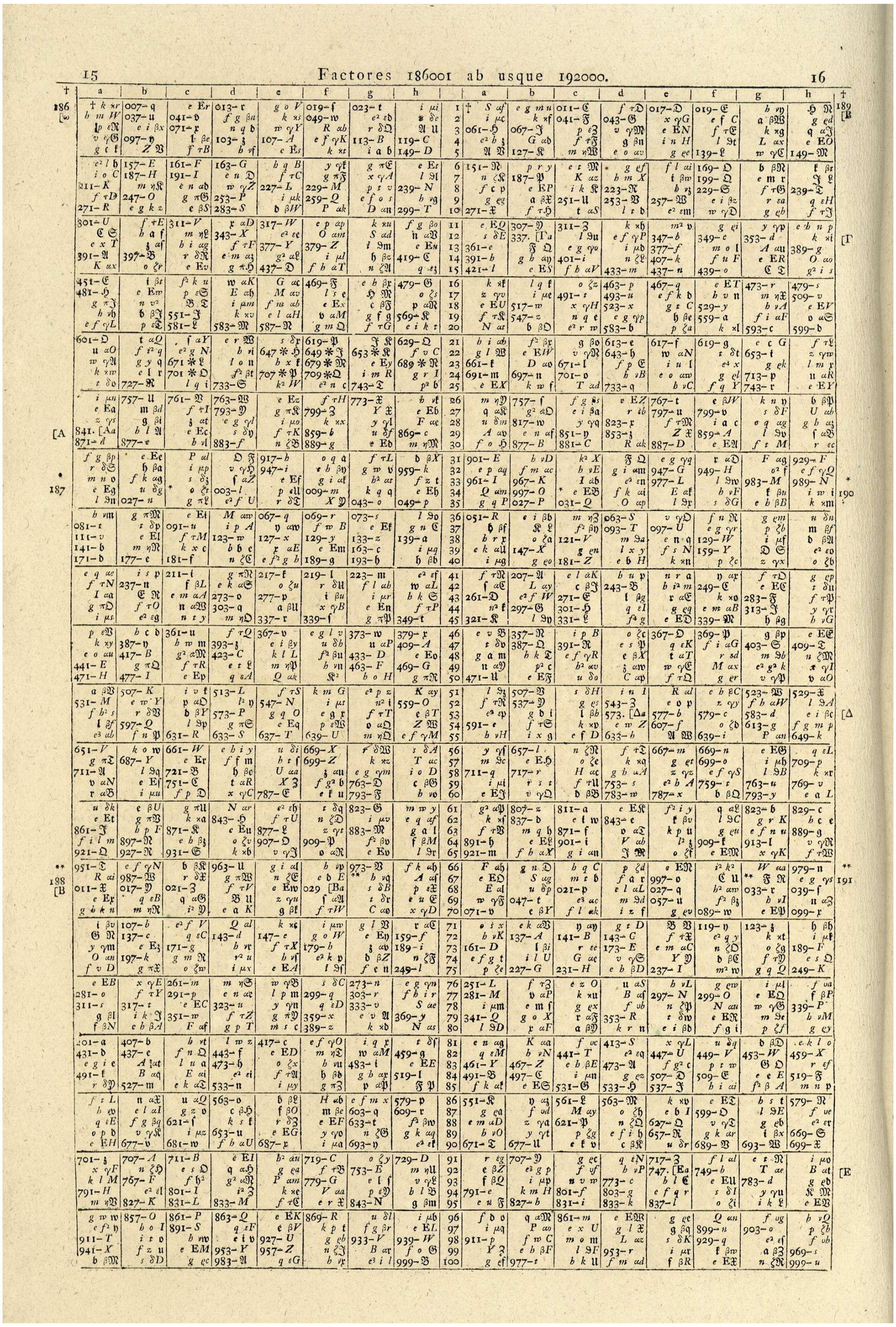


B. Tafel der auf jeder Seite wiederkehrenden Endesziffern.

†	a	b	c	d	e	f	g	h		a	b	c	d	e	f	g	h	†
000	†01	007	011	013	017	019	023	029	1	†01	007	011	013	017	019	023	029	3000
	031	037	041	043	047	049	053	059	2	031	037	041	043	047	049	053	059	
	061	067	071	073	077	079	083	089	3	061	067	071	073	077	079	083	089	
	091	097	101	103	107	109	113	119	4	091	097	101	103	107	109	113	119	
	121	127	131	133	137	139	143	149	5	121	127	131	133	137	139	143	149	
	151	157	161	163	167	169	173	179	6	151	157	161	163	167	169	173	179	
	181	187	191	193	197	199	203	209	7	181	187	191	193	197	199	203	209	
	211	217	221	223	227	229	233	239	8	211	217	221	223	227	229	233	239	
	241	247	251	253	257	259	263	269	9	241	247	251	253	257	259	263	269	
	271	277	281	283	287	289	293	299	10	271	277	281	283	287	289	293	299	
	301	307	311	313	317	319	323	329	11	301	307	311	313	317	319	323	329	
	331	337	341	343	347	349	353	359	12	331	337	341	343	347	349	353	359	
	361	367	371	373	377	379	383	389	13	361	367	371	373	377	379	383	389	
	391	397	401	403	407	409	413	419	14	391	397	401	403	407	409	413	419	
	421	427	431	433	437	439	443	449	15	421	427	431	433	437	439	443	449	
	451	457	461	463	467	469	473	479	16	451	457	461	463	467	469	473	479	
	481	487	491	493	497	499	503	509	17	481	487	491	493	497	499	503	509	
	511	517	521	523	527	529	533	539	18	511	517	521	523	527	529	533	539	
	541	547	551	553	557	559	563	569	19	541	547	551	553	557	559	563	569	
	571	577	581	583	587	589	593	599	20	571	577	581	583	587	589	593	599	
	601	607	611	613	617	619	623	629	21	601	607	611	613	617	619	623	629	
	631	637	641	643	647	649	653	659	22	631	637	641	643	647	649	653	659	
	661	667	671	673	677	679	683	689	23	661	667	671	673	677	679	683	689	
	691	697	701	703	707	709	713	719	24	691	697	701	703	707	709	713	719	
	721	727	731	733	737	739	743	749	25	721	727	731	733	737	739	743	749	
	751	757	761	763	767	769	773	779	26	751	757	761	763	767	769	773	779	
	781	787	791	793	797	799	803	809	27	781	787	791	793	797	799	803	809	
	811	817	821	823	827	829	833	839	28	811	817	821	823	827	829	833	839	
	841	847	851	853	857	859	863	869	29	841	847	851	853	857	859	863	869	
	871	877	881	883	887	889	893	899	30	871	877	881	883	887	889	893	899	
	901	907	911	913	917	919	923	929	31	901	907	911	913	917	919	923	929	
	931	937	941	943	947	949	953	959	32	931	937	941	943	947	949	953	959	
*	961	967	971	973	977	979	983	989	33	961	967	971	973	977	979	983	989	*
1000	991	997	'01	003	007	009	013	019	34	991	997	'01	003	007	009	013	019	4000
	021	027	031	033	037	039	043	049	35	021	027	031	033	037	039	043	049	
	051	057	061	063	067	069	073	079	36	051	057	061	063	067	069	073	079	
	081	087	091	093	097	099	103	109	37	081	087	091	093	097	099	103	109	
	111	117	121	123	127	129	133	139	38	111	117	121	123	127	129	133	139	
	141	147	151	153	157	159	163	169	39	141	147	151	153	157	159	163	169	
	171	177	181	183	187	189	193	199	40	171	177	181	183	187	189	193	199	
	201	207	211	213	217	219	223	229	41	201	207	211	213	217	219	223	229	
	231	237	241	243	247	249	253	259	42	231	237	241	243	247	249	253	259	
	261	267	271	273	277	279	283	289	43	261	267	271	273	277	279	283	289	
	291	297	301	303	307	309	313	319	44	291	297	301	303	307	309	313	319	
	321	327	331	333	337	339	343	349	45	321	327	331	333	337	339	343	349	
	351	357	361	363	367	369	373	379	46	351	357	361	363	367	369	373	379	
	381	387	391	393	397	399	403	409	47	381	387	391	393	397	399	403	409	
	411	417	421	423	427	429	433	439	48	411	417	421	423	427	429	433	439	
	441	447	451	453	457	459	463	469	49	441	447	451	453	457	459	463	469	
	471	477	481	483	487	489	493	499	50	471	477	481	483	487	489	493	499	
	501	507	511	513	517	519	523	529	51	501	507	511	513	517	519	523	529	
	531	537	541	543	547	549	553	559	52	531	537	541	543	547	549	553	559	
	561	567	571	573	577	579	583	589	53	561	567	571	573	577	579	583	589	
	591	597	601	603	607	609	613	619	54	591	597	601	603	607	609	613	619	
	621	627	631	633	637	639	643	649	55	621	627	631	633	637	639	643	649	
	651	657	661	663	667	669	673	679	56	651	657	661	663	667	669	673	679	
	681	687	691	693	697	699	703	709	57	681	687	691	693	697	699	703	709	
	711	717	721	723	727	729	733	739	58	711	717	721	723	727	729	733	739	
	741	747	751	753	757	759	763	769	59	741	747	751	753	757	759	763	769	
	771	777	781	783	787	789	793	799	60	771	777	781	783	787	789	793	799	
	801	807	811	813	817	819	823	829	61	801	807	811	813	817	819	823	829	
	831	837	841	843	847	849	853	859	62	831	837	841	843	847	849	853	859	
	861	867	871	873	877	879	883	889	63	861	867	871	873	877	879	883	889	
	891	897	901	903	907	909	913	919	64	891	897	901	903	907	909	913	919	
	921	927	931	933	937	939	943	949	65	921	927	931	933	937	939	943	949	
**	951	957	961	963	967	969	973	979	66	951	957	961	963	967	969	973	979	**
	981	987	991	993	997	999	**3	009	67	981	987	991	993	997	999	**3	009	5000
2000	011	017	021	023	027	029	033	039	68	011	017	021	023	027	029	033	039	
	041	047	051	053	057	059	063	069	69	041	047	051	053	057	059	063	069	
	071	077	081	083	087	089	093	099	70	071	077	081	083	087	089	093	099	
	101	107	111	113	117	119	123	129	71	101	107	111	113	117	119	123	129	
	131	137	141	143	147	149	153	159	72	131	137	141	143	147	149	153	159	
	161	167	171	173	177	179	183	189	73	161	167	171	173	177	179	183	189	
	191	197	201	203	207	209	213	219	74	191	197	201	203	207	209	213	219	
	221	227	231	233	237	239	243	249	75	221	227	231	233	237	239	243	249	
	251	257	261	263	267	269	273	279	76	251	257	261	263	267	269	273	279	
	281	287	291	293	297	299	303	309	77	281	287	291	293	297	299	303	309	
	311	317	321	323	327	329	333	339	78	311	317	321	323	327	329	333	339	
	341	347	351	353	357	359	363	369	79	341	347	351	353	357	359	363	369	
	371	377	381	383	387	389	393	399	80	371	377	381	383	387	389	393	399	
	401	407	411	413	417	419	423	429	81	401	407	411	413	417	419	423	429	
	431	437	441	443	447	449	453	459	82	431	437	441	443	447	449	453	459	
	461	467	471	473	477	479	483	489	83	461	467	471	473	477	479	483	489	
	491	497	501	503	507	509	513	519	84	491	497	501	503	507	509	513	519	
	521	527	531	533	537	539	543	549	85	521	527	531	533	537	539	543	549	
	551	557	561	563	567	569	573	579	86	551	557	561	563	567	569	573	579	
	581	587	591	593	597	599	603	609	87	581	587	591	593	597	599	603	609	
	611	617	621	623	627	629	633	639	88	611	617	621	623	627	629	633	639	
	641	647	651	653	657	659	663	669	89	641	647	651	653	657	659	663	669	
	671	677	681	683	687	689	693	699	90	671	677	681	683	687	689	693	699	
	701	707	711	713	717	719	723	729	91	701	707	711	713	717	719	723	729	
	731	737	741	743	747	749	753	759	92	731	737	741	743	747	749	753	759	
	761	767	771	773	777	779	783	789	93	761	767	771	773	777	779	783	789	
	791	797	801	803	807	809	813	819	94	791	797	801	803	807	809	813	819	
	821	827	831	833	837	839	843	849	95	821	827	831	833	837	839	843	849	
	851	857	861	863	867	869	873	879	96	851	857	861	863	867	869	873	879	
	881	887	891	893	897	899	903	909	97	881	887	891	893	897	899	903	909	
	911	917	921	923	927	929	933	939	98	911	917	921	923	927	929	933	939	
	941	947	951	953	957	959	963	969	99	941	947	951	953	957	959	963	969	
	971	977	981	983	987	989	993	999	100	971	977	981	983	987	989	993	999	

X

Figure 4: The basic layout of a page. (source: Göttinger Digitalisierungszentrum)



15 Factores 186001 ab usque 192000. 16

Figure 5: A page of Felkel's table. (source: Göttinger Digitalisierungszentrum)

4 Felkel's machine

In order to compute the factors of his table, Felkel made use of a "machine" with which he could work faster. Felkel himself never described this machine in sufficient detail, although it is depicted on the cover of the first part of his table (figure 6).

An advertisement for this machine was published by Wilhelm Bauer in the *Leipziger Zeitung* on 29 July 1776, and in final notes of the fifth volume of Lambert's correspondence, Bernoulli writes that Felkel once attributed the invention of his machine to Wilhelm Bauer, but that it is not clear how much of it is true [15, p. 236]. The announcement by Bauer was mentioned by Hindenburg in the letter he sent on 3 August 1776 to Lambert.

The machine was described in more detail in the review published in the *Allgemeine deutsche Bibliothek* in 1778 [3] and which was reproduced by Bernoulli in the second appendix of his correspondence of Lambert [15, p. 232–234]. Recently, Weiss made a reconstruction of this "machine" which should help clarify its principle [76]. It appears that Felkel used rods. A set of 30 rods was needed to find the multiples of all numbers, and 8 were needed to find the multiples of the numbers not divisible by 2, 3, or 5. These numbers correspond to the possible remainders q when a number N is written in the form $30p + q$. The 8 values correspond to the 8 columns in Felkel's table.

Felkel printed 30 columns of numbers on paper, starting with 1 to 30. Of these columns, 8 were in red.

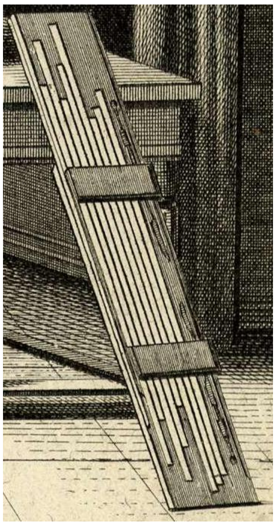


Figure 6: Felkel's machine as it appears on the frontispice of Felkel's table (excerpt of figure 1).

The numbers in each column increased by 30. The 7th rod, for instance, had the numbers 7, 37, 67, 97, until 2977. The thousands were not indicated. Each column had 100 numbers. The paper was glued on cardboard, and cut, and this gave the rods. If we want to find the multiples of 47 not divisible by 2, 3, and 5, we take the rod marked with 47 in the upper third. Then, we multiply 47 by 7, 11, 13, 17, 19, 23, and 29, giving

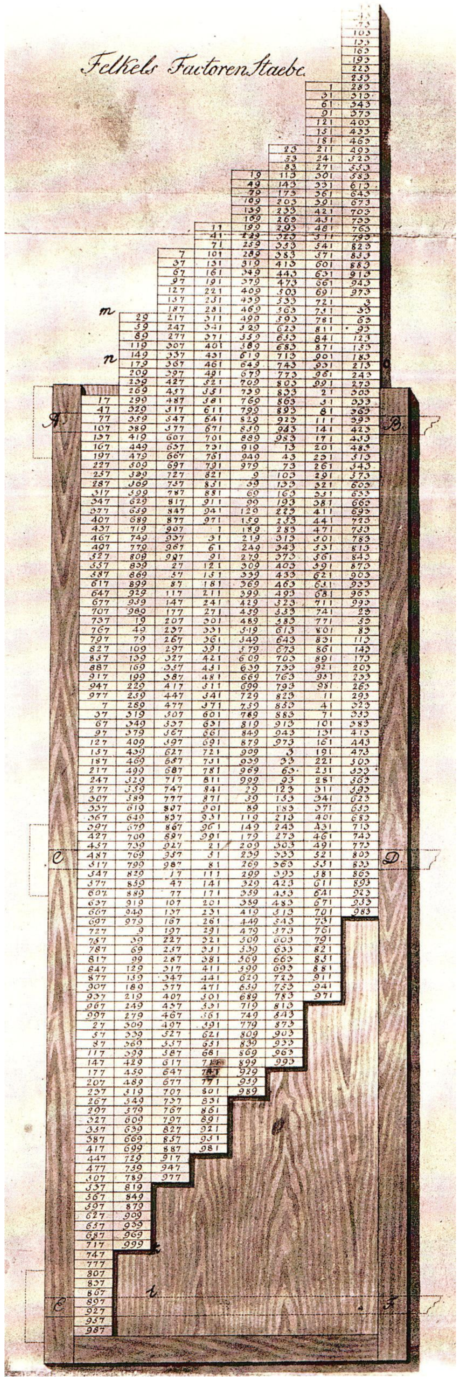


Figure 7: Felkel's machine, as illustrated by Bischoff [16] (courtesy: Stephan Weiss)

329, 517, …, 1081, and 1363. The first product is 329, and we take the rod marked with 329 in the upper third. This is the rod marked with 29, 59, 89, 119, 149, 170, 209, 239, 269, 299, 329, etc. The corresponding rods are taken for the other multiples of 47. For the last two products, 1081 and 1363, we take the rods marked with 81 and 363 in the middle part.

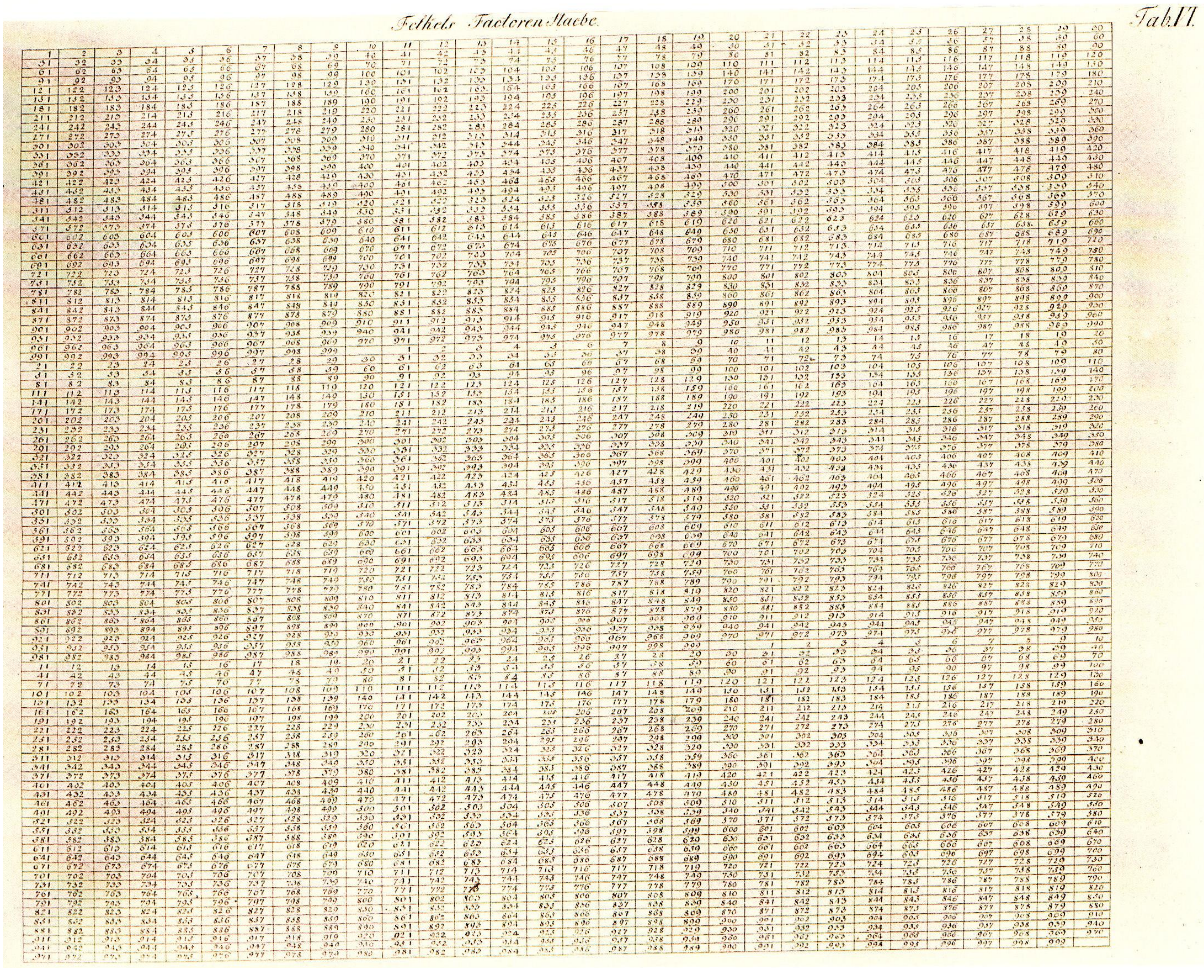


Figure 8: The 30 rods of Felkel's machine, as illustrated by Bischoff [16] (courtesy: Stephan Weiss)

All these rods are put next to each other, and after the first rod, and we now have on a same line the numbers 47, 329, 517, etc., that is 47, 47×7, 47×11, 47×13, …, 47×29. The rods are put on a board and are kept in that arrangement. This is shown on the cover of Felkel's table, for instance. A ruler can be put over this line of numbers.

The next multiple of 47 is $47 \cdot 31 = 1457$. The value 457 is located in the middle part of the first rod, the one with 47. Corresponding to $*457$ ($*$ for one thousand) on the other rods, we find the values $*739$, $*927$, $**021$, $**209$, $**303$, $**491$, and $**773$. These are the products of 47 by $30+7$, $30+11$, $30+13$, $30+17$, $30+19$, $30+23$, and $30+29$. A ruler can be put at this position.

This procedure is repeated by putting a third ruler at the same distance below the second ruler as the second is below the first (47 lines) and this will give the next multiples. If there is no longer enough space below, the ruler can be moved up by 53 lines, but in general it seems that the first product was computed by hand and the others were found with the ruler. This procedure may provide up to 8 multiples for every ruler position. The symbol for 47 then had to be copied in the corresponding cells of the manuscript.

The author of the review added that he gave a detailed description of the machine, because Felkel's own description was not very clear.[13]

5 Errors in Felkel's table

There are very few erratas for Felkel's table, no doubt because locating errors was particularly difficult. Besides the errors given by Felkel himself in a letter to Lambert dated 22 November 1776 [15, p. 110] and in the first part of his table, we know of Florian Ulbrich (1738–1800) who is said to have found several errors in Felkel's table and as a consequence constructed his own table of factors.[14] We also know that Vega used Felkel's table for his table of factors and primes. In fact, Vega did not make it totally clear if he used Felkel's table for his table of factors published in 1797 [74], but he used it for his table of primes. As explained in the next section, however, the errors found in Vega's table show that he did not copy the table of factors from Felkel.

We give here all the known errors, both of content and of sequencing, the latter having been found while analyzing Felkel's table.

5.1 Errors of content

In 1820, a comparison between Vega's table of factors and Ulbrich's was published, and this revealed a number of errors in Felkel's table [7].[15]

First, a number of errors were found in which Vega gave incorrect decompositions. These errors concern the numbers 27293, 43921, 57103, 82943, 90983, and 93137. The article published in 1820 merely stated that all errors (these and the ones below) appeared also in Felkel's table, but we have examined Felkel's table and found the correct decompositions for these six numbers. This suggests that Vega did not copy the first part

[13]For more background on the mechanization of factor search, see chapter 7 of Williams [77].

[14]Florian Ulbrich was Stiftsherr in Klosterneuburg from 1773 to 1791 [48, p. 164], [70], [35], [4]. He computed a table of factors until 1 071 800. This table was never published. Darnaut *et al.* also mention a table of logarithms made by Ulbrich in 1781 [22, p. 105]. By 1793, the clockmaker and mathematician Cajetano, who was apparently quite familiar with Ulbrich's work, wrote that Ulbrich would soon complete his table to 2 millions and that his method was superior to that of Felkel and Hindenburg [19, pp. 41–44]. (We are grateful to Rudolf Fritsch for drawing our attention to the connection between Cajetano and Ulbrich.) According to [5, 60, 8], Ulbrich had actually computed a table to 1.5 millions and had computed auxiliary tables up to 20 millions. Ulbrich's table of factors is now held at the Austrian National Library (Cod. 10684 and 10685), but we have not yet examined it.

[15]According to Lindner, some of these errors had long been known [58]. It is possible that Lindner used Ulbrich's table for his own table published in 1812 [57], but this remains to be checked.

of his table of factors to 102000 from Felkel, but merely took it from Lambert's table, as we initially understood it.

The other errors concern the list of primes given by Vega, and this list ranges from 102001 to 400031. This list was taken from Felkel's table and the errata published coincides with errors in Felkel's table. These errors are the following.

Felkel (and therefore Vega) gave decompositions for the following numbers which are prime: 148303, 148669, 168559, 168677, 173309, 177347, 189913, 194167, 216569, 232103, 235003, 242639, 247609, 326119, 330509, 331921, 336437, 339671, 357817, 357883, 371299, and 397427.

On the other hand, Felkel and Vega gave the following composite numbers as primes: 148363, 148699, 168647, 168859, 173279, 177377, 194107, 210473, 216599, 232163, 235303, 242939, 247669, 326419, 331927, 336467, 339971, 357853, 358837, 359741, 371269, and 397457.

It will be noticed that these two lists often go in pairs, as a misplacement of one factor naturally led to a complementary error nearby. This is similar to some of the errors observed in Kulik's table [51].

The above lists are probably not complete, as the numbers from 400031 to 408000 in Felkel's table have not been subject to a comparison with Vega's table, and might contain other errors.

5.2 Sequencing errors

In addition to the errors mentioned in the previous section, there are others errors involving the names of the primes. We list here the errors we have found, but we have by no means analyzed the whole table.

5.2.1 First part (1–144000)

- Felkel puts "827-𝔈" in the cell of prime number 74827, but he should have put "827-𝔇". As a consequence, all later primes up to the next sequencing error are misnamed.

- 88741 is named 741-𝔚 but should have been called 741-𝔙 if there were only the 74827 error, and 741-𝔘 if there were no errors at all. So, from 88741 on, we have a shift of two letters.

- It is interesting to note that Bernoulli's example of location of a number is in that range and he was unaware of Felkel's sequencing errors. His example is that of 89371, but if there were no errors, the number 89387 would have been meant [15, p. 230].

- For 101789, Felkel writes "789-𝔷" and for 101797, he writes "797-*𝔷". The asterisk seems to mean that Felkel is bringing the shift back to one unit by introducing a variant letter. Such deviations appear several times, and the most likely explanation seems to be that Felkel compared his list of primes with another one, probably that of Marci published in 1772 and which went up to 400000 [59].

- Felkel gave 112699 = 251 × 449 as prime, but he gives the correct decomposition in his errata. Then, for 112799 which is prime, Felkel gives a decomposition, and this error happens to reduce the shift to 0, so that from then on until 144000, the names of the primes are correct (except possibly if two errors cancel out within some pages that we haven't checked). This error in 112799 was also corrected the errata in which Felkel replaced the content of the cell by "799-*" (with no adjoined symbol).

5.2.2 Second part (144001–336000)

After 144000, in the second part of the table, there are also several errors, which are easy to identify by comparing our reconstruction with Felkel's table. We have located the following ones:

- 144773 and 160709 are indicated as prime, but Felkel skips one letter in each case, and he has therefore by then a discrepancy of two letters.
- 168673 and 168697 are given as prime successors (and are not), but Felkel skipped two letters in the naming.
- 174067 was named irregularly as "067*$\mathfrak{u}$" to distinguish it from the previous value "061-$\mathfrak{u}$".
- At the end of the interval 174001–180000, Felkel's count of primes is in excess by 4 units.
- Felkel corrects his count for 182123 which is named "123*k" and at 186000 the excess is only three units.
- The greatest rectification occurs in the interval 186001–192000 (figure 9). Here, Felkel corrects the count by nine units between 186647 and 186709. This is a sequence of nine consecutive primes. As a result, there is now a shortage of 6 units in the count of primes.
- Felkel gave 189913 as composed, but it is a prime. This error was given in [7]. The shortage is now seven units.
- 193031 was named "031-m", like the previous prime "013-m". The shortage is now eight units.
- Felkel skipped one letter for the prime 201473, bringing the shortage back to seven units.
- Other errors occur and eventually the shortage ends with eleven units by 336000.

5.2.3 Third part (336001–408000)

In the third part, the shortage starts with 11 units, and jumps from 11 for 342000 to 16 units for 345000. Eventually, at the end of the last printed page of Felkel's table, for the interval 402001–408000, there is a shift of 13 units between Felkel's numbering of primes, and the exact one. This means that if the problems were only omissions of primes, Felkel would have forgotten 13 primes. In reality, as we have seen, things are a little more complex, since some numbers have been given as primes when they were not, there were some sequencing errors, and there were some intentional shifts, presumably to catch up errors found by comparison with another table.

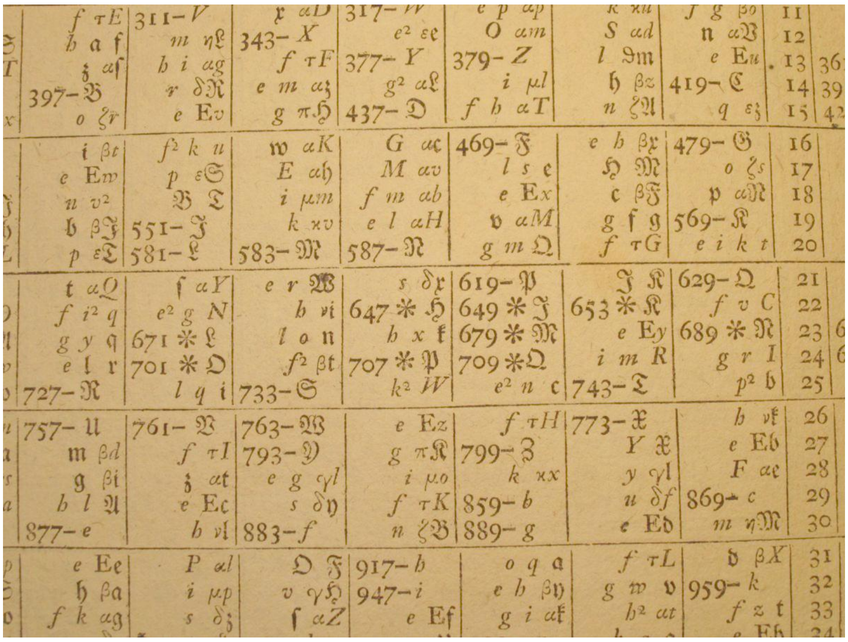


Figure 9: A string of anomalies probably corresponding to errors that Felkel wanted to rectify on the page for the interval 186001 to 192000. The anomalies are the use of '*' in the names of the primes. $649{*}\mathfrak{J}$, for instance, repeats the naming of 551-$\mathfrak{J}$ and therefore shifts the letters. Here, no less than nine symbols are inserted between 629-$\mathfrak{Q}$ and 727-$\mathfrak{N}$.

6 Felkel's later work

Apparently Felkel stopped working on his table of factors in 1777, and took up again the research in 1784 when the correspondence with Lambert began to be printed [15, p. 122], [39, p. 118].

6.1 New computation (1784)

In 1784, Felkel published a tract about a new projected table, with a first part going to 1 008 000, a second part going to 2 millions and with plans to reach 10 millions [15, pp. 121–134]. Moreover, in the preface of the 1798 translation of Lambert's *Zusätze*, Felkel wrote that he computed the table anew from 1 to $2\,856\,000 = 7 \times 408000$ [56, preface, p. VII]. He actually probably computed it up to 3 millions [39, pp. 120–121].[16] It is possible that he had to recompute everything if he no longer had the initial calculations made in 1776.

The new table was going to be very similar to the one published in 1776, the main difference being that Felkel no longer gave all the factors. Felkel now recognized that the use of letters is inconvenient and he proposed to give the smallest factor explicitely, as suggested by Lambert. More precisely, if a number is composed, all the factors except the largest one are given. The first factor is given explicitely, but the other factors are given with letters.

With these new restrictions, for the numbers up to 2 millions, and given that the smallest explicit factor is 7, Felkel only needed symbols for primes up to $\sqrt{\frac{2\,000\,000}{7}} = 534.5$.

The largest prime below this limit is 523. This is also true if the table goes up to 2 016 000 and Felkel was therefore able to use the first column of primes from his first table, where the primes 1 to 523 were encoded by a single letter.

The primes are encoded as in the first table from 1776 (figure 2), namely with a prefix and a letter from the roman or gothic alphabet. Figure 10 shows several primes, for instance on the first line, "$-$M", whose complete name is "($\mu$$-$M", and corresponds to 1 006 507. The prefixes are "(μ," "(ν," and the first appearance of the prefix "(ν" is in column b. A comparison with our reconstruction and extension of the 1776 table will show that the beginnings of the prefixes differ from ours. It is moreover not clear which sequence of prefixes was used by Felkel in this new table. The prefix "(μ" was used in the first table for the primes 59359 to 60493. One may be tempted to include the dot in the prefix (which would then be "(μ."), but the dot does not seem to appear in the first use of the prefix "(ν", so that we will leave this problem open for the moment.

Another difference with the first table is that each page now only covers an interval of 3000 integers, instead of 6000 initially. In the table published in 1776, each page was made of two columns, each covering 3000 integers (figure 4). Each page contained exactly 100 lines. In the new project, only one column was taken, and split in two parts, so that each page now has only 50 lines, and the first line contains the suffixes 001, 007, 011, 013,

[16] Felkel actually wrote that he computed the table from 1 to 5 millions [56, introduction, p. XIV], but Glaisher views this as a misprint, and believes that Felkel did not go beyond three millions [39, pp. 108, 120].

017, 019, 023, 029, and then 501, 507, 511, 513, 517, 519, 523, and 529. This appears clearly on figure 11. The number 1 007 539, which was given in 1785 as an example by Felkel in his article on periods of fractions [34], appears towards the bottom of column f in figure 10 and its factors except the largest 97 are $13.hq = 13 \times 17 \times 47$.

The entire table to 2 016 000 would have occupied 672 pages. Pursuing this scheme to 10 millions would have required 3334 pages, but then Felkel would not have been able to use only one-letter symbols.

Felkel again started a subscription. He planned to first issue the first million, then installments of 200000 four times a year. But Felkel's project apparently failed again, perhaps for lack of time on his side, or because he moved to Lisbon. His project was only fulfilled on a small scale in Felkel's translation of Lambert's table in 1798.

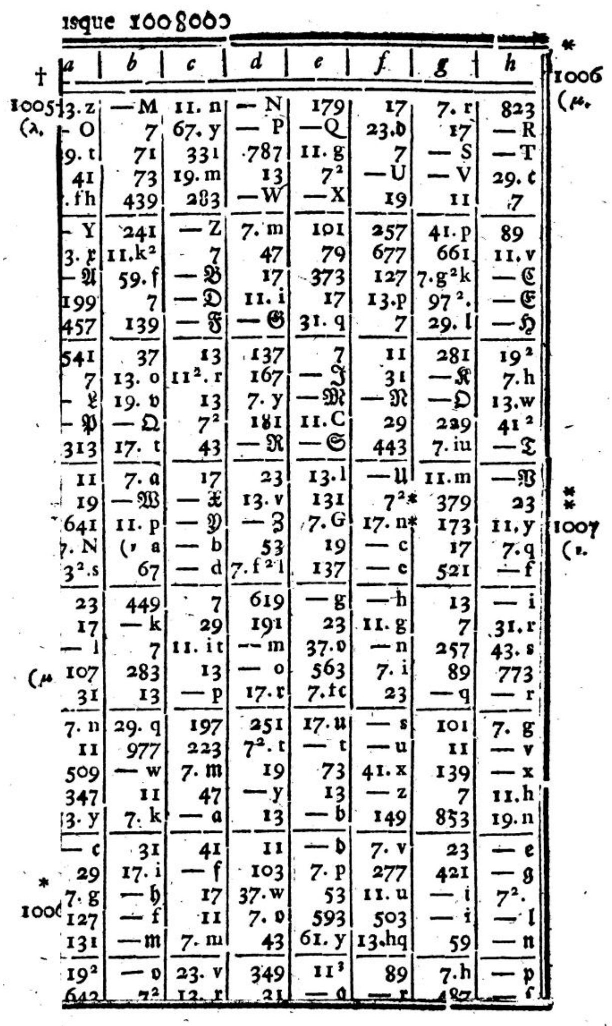


Figure 10: A fragment of the last page of the first part of the new projected table (1784). The page was scanned by Google without unfolding the table, so that the left part (with another set of columns a to h) is not visible, except for the marginal inscriptions 1005 and 1006. The lower part of the table is also not visible. Part of it is visible in figure 11.

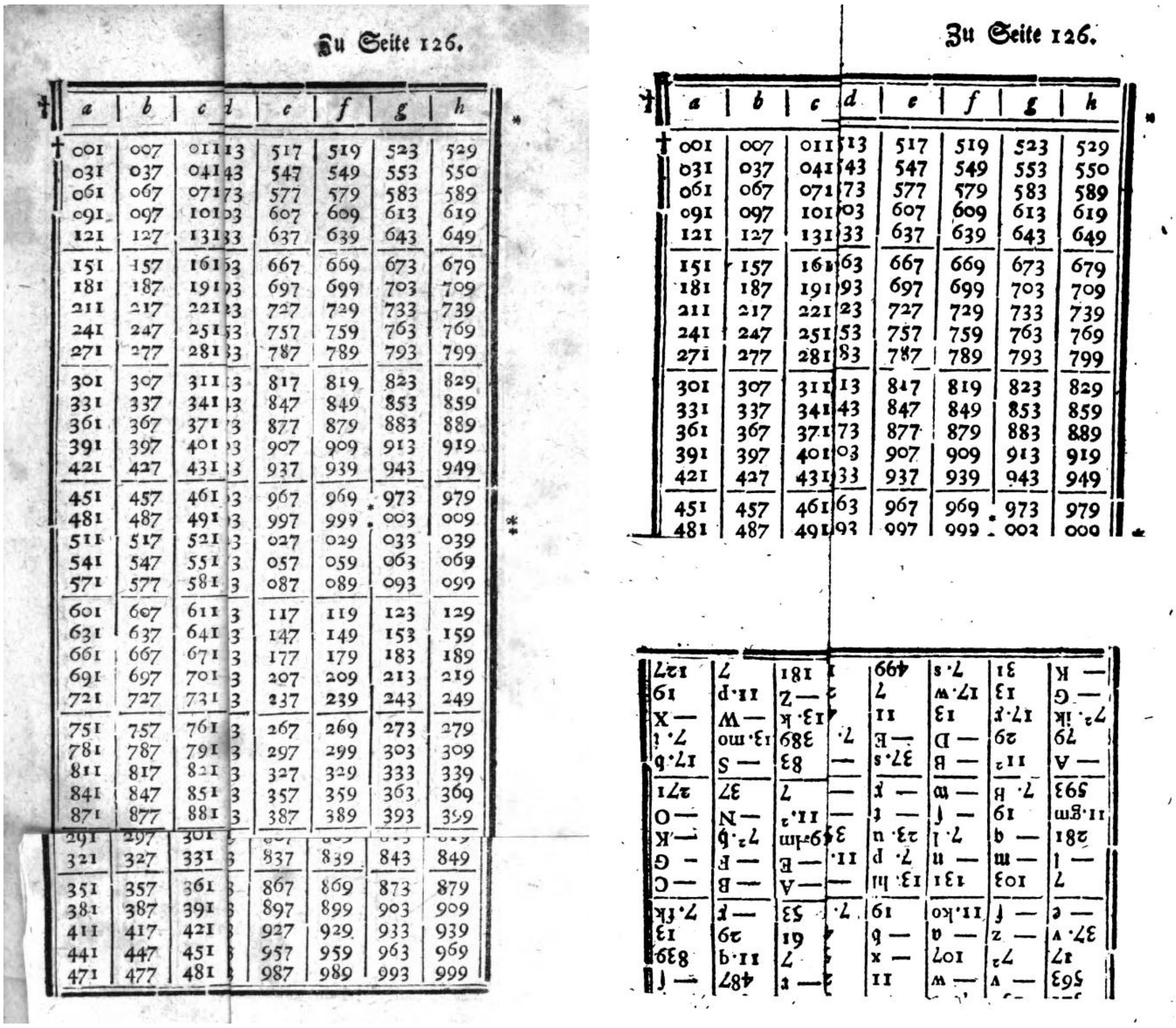


Figure 11: The layout of a page in the 1784 project. These are two different scans, but in both cases the page was scanned by Google without unfolding the table. On the right scan, we can see the bottom of the last page of the projected table, but folded.

6.2 Felkel's new method (1793–1794)

In the preface of the 1798 translation of Lambert's table, Felkel writes that he was working on a more portable scheme and he constructed 15 "bases" by the aid of which numbers up to 24.6 millions could be factored, when the least factor does not exceed 400 [56, preface, p. VII], [39, p. 121]. Felkel also found a method to factor numbers up to 100 millions on 65 "bases." Zach also mentioned a table up to 24 600 000 [75].

There is an account of Felkel's new method in a letter sent by Stockler to Felkel [36], probably around 1795. In order to decompose the numbers from 40000 to 1 200 000, Felkel proceeds as follows. He first assumes that he has the decompositions of all integers up to 40000 and he considers a table of the numbers $30n+1$, $30n+7$, $30n+11$, $30n+13$, $30n+17$, $30n+19$, $30n+23$, $30n+29$, for $n < 1200$. Now, if $N = 30n + a$ is a number to decompose, Felkel considers the double sequence

$$\begin{array}{cccccccc} a & 30+a & 60+a & 90+a & \ldots & X & \ldots \\ n & n-1 & n-2 & n-3 & \ldots & Y & \ldots \end{array}$$

If X and Y have a common factor, then it is easy to show that this is also a common factor of N. For instance, for $N = 40003 = 1333 \times 30 + 13$, we have the double sequence

$$\begin{array}{cccccc} 13 & 43 & 73 & \ldots & 763 & \ldots \\ 1333 & 1332 & 1331 & \ldots & 1308 & \ldots \end{array}$$

and 763 and 1308 have the common factor 109, which divides $40003 = 109 \times 367$. The common factors are found using the known decompositions until 40000, and this sets the limit of n to 40000, and therefore of N to 1 200 000. This technique can be extended for larger intervals.

The bases alluded to by Felkel are apparently tables of numbers not divisible by 2, 3, or 5 in certain ranges. For instance, Felkel calls "base A" the table of the numbers from 1 to 36000 not divisible by 2, 3, or 5. "Base B" contains the numbers not divisible by 2, 3, or 5 from 2 364 000 to 2 400 000 [36, p. 392].

6.3 Felkel's edition of Lambert's table (1798)

Although Lambert had a difficult time with Felkel, as testified by Lambert's correspondence, Felkel somehow had his revenge in publishing a Latin translation of Lambert's *Zusätze* with an improved table of factors in 1798 [56]. This table was clearly influenced by Felkel's 1784 table, in that Felkel also tried to give all the factors, except the largest one.

7 Chronology 1770–1784

The period between the publication of Lambert's tables in 1770 and the publication of Felkel's table and Hindenburg's method was a period of intense development and uncompleted computations which must have been a source of frustration for all the involved parties.

Thanks to the publication of Lambert's correspondence by Johann Bernoulli,[17] we have a fairly detailed picture of these developments. The correspondence contains in particular the letters exchanged with Felkel and other table computers. Bernoulli organized the letters by correspondents, and then by date. In this section, we have summarized the parts of each letter relevant to the computation of factor tables, and we have ordered them strictly chronologically. This seemed adequate, since there are often allusions to one correspondence in another, and the development and interactions between Lambert, Felkel and others can be followed more easily that way.[18] We have also added a few other events in this chronology.

The chronology starts in 1770, after the publication of Lambert's table [54].

- On 27 September 1770, Lambert publishes a notice in the *Allgemeine deutsche Bibliothek* on the tables that he received from Oberreit, without naming him [55]. Oberreit has extended the table to 150000, and plans to extend it at least to 204000. Lambert published this notice in order to avoid duplicate work. Lambert also mentions a second computer, but it is not clear if he refers to Wolfram or someone else.

- In a letter sent to Kant in 1770 [12, p. 366],[19] Lambert writes that he found someone to extend his table of factors to 204000. He writes that the table of factors will perhaps be extended to one million. (In 1782, Bernoulli adds that this table to 204000 was computed by Oberreit,[20] that it was extended to 500000, and that it was now in the hands of professor Schulze in Berlin. Bernoulli also mentions a work giving the natural logarithms of all prime numbers up to 100000 [12, p. 368].)

- 5 March 1772, Wolfram[21] to Lambert [14, p. 436]

  Wolfram writes that he received Lambert's 1770 book on February 28, and he informs Lambert that he had computed for his personal use a table giving the smallest factor of all numbers up to 300000 not divisible by 2, 3, and 5. This table occupies 25 pages. (see also [15, p. 140])

[17] Johann III Bernoulli (1744–1807).

[18] Lambert's correspondence about tables of factors was also described by Glaisher [39] and Bullynck [17, 18].

[19] This letter is also reproduced in Kant's published correspondence [45].

[20] Ludwig Ober(r)eit was an Ober-Finanz-Buchhalter in Dresden. He was born in Lindau in 1734 and died in Dresden in 1803. He was the brother of Jacob Hermann Ober(r)eit (1725–1798) who rediscovered one of the main manuscripts of the Nibelungenlied. See also *Neue deutsche Biographie*, volume 19, 1999, pp. 382–384.

[21] Isaac Wolfram. A summary of what is known on Wolfram was given by Archibald [10].

- 21 March 1772, Lambert to Wolfram [14, p. 441]

  Lambert mentions Oberreit who planned to extend the table of factors to one million, giving all factors. Lambert writes that he published a notice about Oberreit's computations on 27 September 1770, in order to prevent duplicate work.

- 3 August 1772, Wolfram to Lambert [14, p. 445]

  Wolfram mentions the work of Adolph Marci who had factored the numbers up to 300000 by 1756. Wolfram had known about this table since 1758. Wolfram also had plans to extend the table to a million in 1756.

- 5 April 1773, Wolfram to Lambert [14, p. 493]

  Wolfram mentions the table "Tafelen over de Primgetallen van 1 tot 400000" published by Marci in 1772 or 1773 [59].

- 26 July 1773, Wolfram to Lambert [14, p. 497]

  Among other things, Wolfram writes that among the 266666 numbers smaller than one million and having their smallest factor greater than 5, there are 78458 primes, 121852 numbers with two prime factors, 55244 numbers with three prime factors, 10160 numbers with four prime factors, 907 numbers with five prime factors, 44 numbers with six prime factors, and only one number with seven prime factors.[22]

- 18 May 1774 : Lambert to von Stamford [15, p. 5]

  Lambert mentions to von Stamford[23] that the table of factors had been computed from 1 to 72000 and from 100000 to 504000. This is the work done by Oberreit.

- 24 May 1774: von Stamford to Lambert [15, p. 8]

  von Stamford doesn't understand why there is a gap between 72000 and 100000, given that there is none in Lambert's 1770 table. He asks for clarifications to Lambert. In fact, in 1770, Lambert gave only the smallest factors, whereas Oberreit gave all factors.

- 7 June 1774: Lambert to von Stamford [15, p. 10]

  Lambert explains how to fill the gap between 72000 and 100000.

- Hindenburg[24] started to work on the tables in 1774 [15, pp. 156, 240].

- 11 August 1774, Lambert to Wolfram [14, p. 509]

[22] This number is $7^7 = 823543$.

[23] Heinrich Wilhelm von Stamford (1740–1807), officer in Potsdam. From 1772 to 1775, he was teaching French at the monastic school of Ilfeld in Germany [62].

[24] Carl Friedrich Hindenburg (1741–1808) developed the combinatorial school [21].

Lambert mentions the work of von Stamford. Lambert received from Oberreit the factors of the numbers from 1 to 72000 and from 100000 to 500000,[25] with the notice that Oberreit could no longer complete it. von Stamford is now working on filling this gap.

- 2 April 1775 : von Stamford to Lambert [15, p. 15]

  von Stamford announces that he filled the gap from 72000 to 100000. He announces that he agreed with Rosenthal[26] to split the computation from 504000 to one million. von Stamford will take the range 504000 to 750000 and Rosenthal will go from 750000 to one million.

- 12 August 1775 : Lambert to von Stamford [15, p. 20]

  Lambert writes that the limit should be pushed to 1002000 in order to fill the last page.

- 2 December 1775, von Schönberg to Lambert

  Curt Friedrich von Schönberg, a student of Hindenburg, writes that Hindenburg has been working on factor tables and that his method would allow him to recompute Lambert's table to 102000 in 8 days [13, p. 310].

- 26 December 1775, Lambert to von Schönberg

  Lambert gives a list of tables he already has, and he mentions the table giving all factors to 500000, and that he had a pledge that this table would be extended to one million [13, p. 312].

- 15 January 1776: Felkel to Lambert [15, p. 33]

  Felkel announces the completion of his table to 144000, on 12 sheets (24 pages, with a range of 6000 per page). He says that he tried to build such a table, without knowing if such a table did already exist. He writes that he spent six days to find all factors of odd numbers from 1 to 10000, and it occured to him that the limits of 100000, or even 1 million, were not unreachable.

  On one side of a half-sheet, he put an interval of 3000 numbers, and he had such a page printed in 400 copies. We can therefore assume that he had pages for $400 \times 6000 = 2\,400\,000$ numbers. He then describes his first sieve, a "scale that could be turned into a machine."

  The interval of 144000 took Felkel two months and he hoped to reach 10 millions in several years. He wrote of 840 sheets, that is 1680 pages, which is the number

[25] In fact, probably 504000. According to [15, p. 140], Oberreit had completed the table up to 260000 by the Summer 1771, and the next year he had reached 339000, both having been sent to Lambert. In 1775, and perhaps earlier, the table was ready to be printed until 500000 (actually 504000) and the plan was to go up to a million. There was however apparently a gap between 72000 and 100000 (or 102000?).

[26] Gottfried Erich Rosenthal (1745–1813) was a pioneer of meteorology and measuring instruments. He was *Bergkommissarius*.

of pages if the initial scheme is continued. Felkel was looking for the support of Lambert and the Academy.

The numbers with factors 2, 3, and 5 were not included in Felkel's table.

Felkel briefly mentions that he provides two sieves, one for numbers not divisible by 2, 3, and 5, and which provides the factors of 8 numbers for every move; the second sieve applies to all numbers and identifies the factors of 30 numbers at a time.

- 3 February 1776, Felkel to Lambert [15, p. 44]

  In this letter, Felkel advertises the making of the table until one million by subscription, on 84 sheets. He writes that factors can be found 40 times faster than by hand with his machine. Felkel also sells his machines and claims their usefulness for people who would like to find factors of numbers not within his table. Finally, Felkel estimates that he can complete one million every six months and that completing a 10 million table should not be given up.

- 12 February 1776, Lambert to Felkel [15, p. 51]

  Lambert informed Lagrange of Felkel's project, and writes that Lagrange only has an incomplete understanding of Felkel's machine. Lagrange is not convinced that the machine can be used with large numbers.

- 21 February 1776, Felkel to Lambert [15, p. 53]

  Felkel writes that he takes up the task of computing the factors until 2 millions (or beyond). One year should suffice. At that point he had already computed more than half a million. Some errors are mentioned but they probably concern a preliminary version of his table to 144000, for the mentioned errors do not occur in the published version. Felkel writes that he is preparing a description of his machine. Felkel also writes that his work has been examined by 15 mathematicians.

- March 1776, Lambert to Felkel [15, p. 57]

  Lambert encourages the computation from 1 million to 2 millions. Lambert mentions his other two correspondents who were working on the interval from 500000 to 1 million, namely Stamford and Rosenthal. Another correspondent, Ludwig Oberreit, had extended the table to 500000. Lambert writes that he does not object the publication of Felkel's table to 144000 which will find some use, but he questions Felkel's choice of a subscription for the table. But at the same time, Lambert thinks that a subscription is the best way to sell the machine.

- February or March 1776: Lambert to Rosenthal [15, p. 24]

  Lambert writes to Rosenthal because he has had notice of Felkel's work. Lambert says that he advised Felkel to continue with the second million.

  Lambert hopes that the works of each part will fit together.

- 6 March 1776: Rosenthal to Lambert [15, p. 26]

Now, Rosenthal writes that he was informed by Stamford that the latter's illness does no longer allow him to continue the work, and Rosenthal agreed to do all the work himself.

Rosenthal asks Lambert if he can obtain for him a machine by Felkel.

- 30 March 1776, Felkel to Lambert [15, p. 62]

  Felkel writes that he finds the factors of 240 numbers in one hour. He writes that a 10 year old child could operate the machine and dictate its results. He estimates that the numbers from 1 million to 2 millions could be factored in at most 2927 hours, and in fact faster, in particular by the use of several computers.

  Felkel proposes to complete the table until 1 008 000 for Easter 1777, to have it printed on 42 sheets, and to have the factors up to 2 000 000 ready for print. (but shouldn't the first million be on 84 sheets?)

  Felkel writes that it would be good to print the table until 300000, and that it would occupy 12.5 sheets. So, on one sheet, there would be four pages, since $300000 = 12.5 \times 4 \times 6000$.

  Felkel plans to work on parts of 300000 numbers and seven such parts amount to a little more than 2 millions. This would make it easier for Felkel to distribute the work among aides.

- 2 April 1776, Wolfram to Lambert [14, p. 520]

  Wolfram writes that he had factored the numbers from 1 to 126000 in 1743 in Danzig, and continued up to 300000 in Holland. He also gives a list of 30 errors in Marci's table up to 300000. He writes that Marci had planned to go to 1 200 000.

  According to Wolfram, the prime numbers until 100000 in Marci's book are printed like those of Jäger published by Krüger in 1746.

- 24 April 1776, Felkel to Lambert [15, p. 70]

  Felkel explains that he is now compelled to publish a table in order to satisfy his patrons. The printing will start in May 1776. This may explain why the table to 144000 was printed.

  Felkel observes that he computes the tables as fast as the printer can print them, and he works without any aide. Felkel is confident that by the following Easter, the first two millions will be computed, and a large part of the second million will be printed.

- 24 Juni 1776, Felkel to Lambert [15, p. 73]

  Felkel writes that his computations go faster than the setting by two printers. Felkel now plans to write an introduction to the first table.

- 30 July 1776: Lambert to Rosenthal [15, p. 27]

  Lambert writes that he was compelled not to answer Felkel's letters, as Felkel is not willing to cooperate with other computers. Lambert writes that he has already three unanswered letters by Felkel. Lambert's letter is quite critical of Felkel's projects and arrogance.

Lambert now suggests to Rosenthal to wait for Felkel's completion of the two millions and for the description of his machine, until further decisions be taken.

- 3 August 1776: Hindenburg to Lambert [15, p. 137]

  Hindenburg informs Lambert of a published announcement to go beyond Lambert's table. Hindenburg mentions a machine advertised by Wilhelm Bauer, professor in Vienna.

  Hindenburg knows that Lambert has already obtained all factors of the numbers up to 500000.[27]

  Hindenburg writes that he plans to provide a million, but only with the smallest factor. He also wants to provide a million, to make sure to extend beyond other competitors.

  This letter contains a printed notice from 24 May 1776 by the publisher Siegfried Leberecht Crusius about Hindenburg's project of giving the smallest factor of the numbers between 1 and 5 millions [15, pp. 142–150].

  Crusius' notice is also an answer to Lambert's call. The notice insists on the purely mechanical and reliable process of finding the primes and factors.

  According to this notice, the computation of the interval from 1 to 102000 took a little less than 14 days. This table contains 27200 number positions.[28] There are 17433 factors and 9767 prime numbers. Hindenburg only had to search the factors of 11427 numbers.

  The notice observes that the way the factors are computed is very regular, and it makes it possible to predict the time of completion of the table. It is also mentioned that Hindenburg would be happy to have others do the computation, if there were an unexpensive way to check the other computers' results. And this way must be easier than computing the factors by himself.

  At the beginning, Hindenburg had planned to issue intervals of 200000 at a time, but he now plans to issue one million at a time.

  For reasons unrelated with the computation, the first million cannot be printed before Easter 1777.

  The layout will follow that of Lambert, with some changes, and there will be two fifths more numbers on a page than on two consecutive pages by Lambert. (There are 800 numbers on two pages in Lambert's arrangement and so there might be about 1100 numbers on one page of Hindenburg's project) It seems that there will be intervals of 5000 numbers per page.

  One octavo sheet will cover an interval of 40000 and 25 sheets will cover a million.

[27] On details of the communication of this table to Lambert, see [15, p. 140].

[28] Each double page contains 800 values and there are 34 double pages, hence $34 \times 800 = 27200$.

- 13 August 1776: Lambert to Hindenburg [15, p. 151]

  Lambert informs Hindenburg of Felkel's projects. He writes that he had already told Felkel that it was not necessary to compute the table from 1 to 1 million, because this had already been computed, and that he could instead compute the table from 1 million to 2 millions. Felkel consequently altered his plans to extend the first table to 300000, and to announce the continuation to 2 millions. Lambert complains that Felkel asked him to support his endeavour, although Felkel did not describe his machine. Consequently, Lambert did not answer the last three letters by Felkel. Lambert now expects Felkel to extend his plans further, since Felkel mentioned the limit of 10 millions in a printed notice.

  Lambert writes that Felkel does not realize that his method and machine could make him more famous than an unnecessary use of them.

  Lambert even writes that before what happens with Felkel becomes clear, Lambert will not publish anything on factors, as he considers that Felkel might claim that he already knew everything better.

  Lambert mentions that such duplicate work occurred before, for instance for the translation of foreign works.

  Felkel is not intimidated by 5 millions, and has already found people to assist him.

  Lambert says that because Felkel does not listen to him, he is happy to see Hindenburg's work progress.

- 13 August 1776: Lambert to Rosenthal [15, p. 30]

  Lambert informs Rosenthal that Hindenburg, after learning from Felkel's proposal, decided to extend the table up to 5 millions, but only indicating the smallest divisors. Lambert says that Hindenburg and Felkel want to compete, instead of cooperating. Lambert concludes that this affair will in the future occupy a certain space in the history of mathematics.

- 13 August 1776, Lambert to Felkel [15, p. 81]

  Lambert mentions the work of Hindenburg, without naming him, after the letter Lambert received of him above (dated 3 August).

- 17 August 1776, Hindenburg to Lambert [15, p. 155]

  Hindenburg estimates that he can compute faster than Felkel with his machine.

  He writes that it seems that Felkel does not follow Lambert's layout. (In a footnote, Bernoulli adds that Felkel followed Euler's arrangement)

  Hindenburg observes that no cooperation is possible with Felkel.

- 24 August 1776: Lambert to Hindenburg [15, p. 161]

  Lambert tells Hindenburg that he only sent to Felkel Hindenburg's printed notice.

  Lambert recalls, without naming them, the works of Oberreit, von Stamford and Rosenthal.

Lambert mentions Euler's 1775 article [26] and his proposed layout.

- 10 September 1776, Felkel to Lambert [15, p. 82]

  In this letter, a conflict with Hindenburg is mentioned for the first time. Felkel writes that he could also promise 20 millions.

  This letter also contains a long "Nachricht", dated 1st June 1776, but actually probably 1 September 1776. (see Hindenburg's letter from 27 September 1776 and also Bernoulli's comment [15, p. 235]; the same text does exist with two different dates) This text announces two tables, one giving all factors of all numbers up to 1 million, and another giving all factors of all numbers not divisible by 2, 3, and 5 from 1 to 10 millions.

  This text also gives the names of 16 "mathematicians" who testified about Felkel's work.[29] Felkel then writes that the first million has been completed for some time. He writes again that one year is enough to compute two millions, assuming that no more than four hours are spent every day on it.

  So, his plan is to compute two millions every year until 10 millions. But he also writes that he now has two computers by his side.

  Felkel recollects the influence of Lambert and that he had started his computations, not knowing that others had also partly extended Lambert's table. Felkel writes that Lambert encouraged Felkel to go beyond the first million.

- 27 September 1776, Hindenburg to Lambert [15, p. 166]

  Hindenburg writes that for priority reasons he had to publish a notice of his method.

  He also mentions a notice by Felkel, dated 1 September 1776. This notice must be the one Felkel sent to Lambert on 10 September. Hindenburg accuses Felkel of having borrowed words from Hindenburg's notice.

  Hindenburg mentions the publication of his "Beschreibung einer ganz neuen Art etc." [42].[30]

  Hindenburg writes again that he puts 5000 numbers per page. He now gives for every divisible number n the smallest factor p, as well as the remaining factor, that is p and n/p. He claims again to find the factors faster than Felkel.

  Hindenburg now also presses Lambert for his opinion on his work.

- 2 October 1776: Hindenburg to Lambert [15, p. 173]

  Hindenburg gives additional details. (Hindenburg's table also does not contain multiples of 2, 3, and 5) He writes that for every million 60000 factors can be printed in advance. (I fail to see why, as one million is not a multiple of any factor that is found in the table) Hindenburg writes that Felkel's machine does not have this advantage, however it is made. Hindenburg also stresses the usefulness of

[29]See Felkel's letter of 21 February, above.

[30]Reviews of Hindenburg's publication appeared in 1777 [2] and 1778 [3]. For a critique of Hindenburg's text, see Knuth [49].

adding the multiplicity of the smallest factor and the co-factor (that is, every non prime number is written as $a^p \cdot b$). The co-factor could be written in italics when it is not a prime number. He writes that this almost as good, and in a certain sense even better than giving all the factors.

- 5 October 1776: Lambert to Hindenburg [15, p. 179]

  Lambert informs Hindenburg that Felkel wrote him that Hindenburg's notice was useful to him.

  Lambert makes some observations on the layout of the table, for instance considering that $3556553 = 7 \cdot 508079 = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. The first decomposition takes a lot of space, but less than the complete one. Lambert also suggests to put an asterisk after a prime number in a decomposition.

- 18 October 1776, Wolfram to Lambert [14, p. 523]

- 8 November 1776, Felkel to Lambert [15, p. 106]

  In this letter, Felkel even writes that his plan would need no modifications until 1000 millions. He writes that he can find all factors of 500 numbers in one hour.[31]

- 22 November 1776, Felkel to Lambert [15, p. 110]

  Felkel mentions someone who, being stimulated by Felkel's work, decided to compute the logarithms of all prime numbers up to a million with 15 decimals.

  Felkel also gives an errata to his table. This errata is for the table to 144000, but does not account for all errors.

- On 29 November 1776, Hindenburg published a notice of his projected table in the *Wittenbergsches Wochenblatt* [1].

- 30 November 1776, Lambert to Wolfram [14, p. 524]

  Lambert mentions Felkel and Hindenburg, but without naming them. He writes that the one in Leipzig (Hindenburg) wants to go to 5 millions, and the one in Vienna (Felkel) to 2, or even to 10 millions. Lambert observes that the tables can be organised in columns of intervals of 300, that is, of 80 values.

- 7 December 1776: Hindenburg to Lambert [15, p. 184]

  Hindenburg now sends his complete 1776 text "Beschreibung ..." [42]

- 10 December 1776: Lambert to Hindenburg [15, p. 188]

  This is an answer to Hindenburg's letter from 2 October, and Lambert had apparently not yet received the letter from 7 December.

  Lambert writes that Felkel sent him his table up to 144000 and he is glad that he will be able to tell him his opinion (and this he did on 12 December).

[31] In 1785, he wrote that he could factor up to 800 numbers hourly [15, p. 492].

Lambert gives a description of Felkel's table. The use of letters is criticized, but I think that Felkel's table is still the only one that could go so far in a compact way. Kulik's table goes beyond, but by giving only the smallest factor.

Lambert says that he will suggest to Felkel giving explicitely the first factor.

Lambert also describes Felkel's machine. It is made of 8 small rods and 30 big ones. Lambert observes that a circle or a cylinder could be used, and even clock or gearwork, which anticipates the work of Lehmer and others.

Lambert now understands that Felkel could not have started at 1 million.

Lambert also writes that Wolfram is still unaware of the latest developments.

- 12 December 1776, Lambert to Felkel [15, p. 114]

  This is Lambert's first critical letter to Felkel. Lambert observes first that laying out the numbers in eight columns $30n+1$, $30n+7$, $30n+11$, $30n+13$, $30n+17$, $30n+19$, $30n+23$, $30n+29$ is a good idea. Lambert understands that the letters are used to save space, but he believes that the factors could have been written in full, without loosing too much space. (I don't think this is true, though.) Lambert also considers that the use of letters creates a separation between Felkel's work and the work of others, so that neither can be helpful to the other.

  Lambert is also somewhat critical of Felkel's machine, writing that it is not so new. Lambert observes that everything reduces to 1) laying out the tables in the best way, and 2) finding in the shortest possible way the positions where any factor is to be inserted. Lambert alludes to some useful formulæ by Wolfram for that purpose.

  Lambert refers to Hindenburg's work in Leipzig, and the publication of his "Beschreibung einer ganz neuen Art etc." [42]. Lambert writes that Hindenburg's work differs a lot from Felkel's and contains a lot of theory.

  He mentions again the person who computed the table to 500000 and another person who also did some work, but they are not named by Lambert.

  Lambert considers that most readers will understand that these tables are useful, but will not consider them necessary to them.

  Regarding the computation of logarithms, Lambert considers that the most important is to have tables without errors, and that it would be good to have a table of logarithms from 990000 to 1 000 000, with their differences.

- 14 December 1776: Lambert to Hindenburg [15, p. 194]

  Lambert now answers Hindenburg's letter from 7 December.

  Having two copies of Hindenburg's treatise, Lambert sent one of them to Lagrange. He also sent him the copy of Felkel's work.

  Among Lambert's observations to Hindenburg, he makes some comments on the decimal periods given in Hindenburg's work (pp. 113–115).

- 22 December 1776: Hindenburg to Lambert [15, p. 199]

  This is the last letter from Hindenburg to Lambert.

  Hindenburg now comments on Felkel's table, which he has also obtained. He observes that Euler had already used the same arrangement as Felkel.

  Hindenburg observes that Wolfram probably lives in a place where few news arrive.

  The rest of the letter is concerned with decimal periods.

- 8 March 1777, Lambert to Wolfram [14, p. 531]

  Bernoulli mentions here an additional note by Lambert with the number of primes in a number of intervals.

- 9 August 1777, Lambert's last letter to Wolfram [14, p. 536]; Lambert died on 25 September 1777.

- Simultaneous accounts of the publication of Hindenburg's method and Felkel's table eventually appeared in 1778 in the *Allgemeine deutsche Bibliothek* [3].

- In 1782, Hindenburg published again a notice about the upcoming publication of his table [43], but the table was never published.

- In November 1783, Felkel published a short notice about his machine [33], in which he also wished that Hindenburg's tables be at least deposited in a library in Leipzig.

- Bernoulli writes that he saw Hindenburg's first million in September 1784 in Leipzig, where Hindenburg was professor [15, p. 242].

8 Conclusion

Felkel's table of factors was one of several interesting projects spawned by Lambert's call for tables. Felkel chose an uneasy road and created a quite unique and beautiful table of factors.

But Felkel's table was little used. As mentioned above, it was chiefly used by Vega as a basis for his table of primes from 102001 to 400031 published in 1797 [74]. But this table was included in the *Sammlung mathematischer Tafeln* edited by Hülße, in 1840 and 1849 [44]. Some of Vega's errors were corrected, but not all of them. $173279 = 241 \times 719$, for instance, is given as prime in both editions, and this error goes back directly to Felkel.

Vega's tables, and later Burckhardt's, made it unnecessary for anyone to delve into the intricacies of Felkel's rare opus, and it is likely that Vega was Felkel's only heir. We can only hope that the wider availability of Felkel's original works and our reconstruction will contribute to a better recognition of his work.

References

The following list covers the most important references[32] related to Felkel's table. Not all items of this list are mentioned in the text, and the sources which have not been seen are marked so. We have added notes about the contents of the articles in certain cases.

[1] Anonymous. (Review of Hindenburg's book). *Wittenbergsches Wochenblatt*, 9(48):385–387, 1776. [issue dated 29 November 1776]

[2] Anonymous. (Review of Hindenburg's book). *Zugabe zu den Göttingischen Anzeigen von gelehrte Sachen*, 1:29–31, 1777.

[3] Anonymous. (Reviews of Hindenburg and Felkel's books). *Allgemeine deutsche Bibliothek*, 33(2):494–497, 1778.

[4] Anonymous. Des Hrn. Florian Ulbrich's regulirten Chorherrn und Stadtpfarrer zu Korneuburg in Niederöstreich. *Archiv für Geographie und Statistik, ihre Hilfswissenschaften u. Litteratur*, 1(3):194–196, 1800. [obituary, reproduced from the Wiener-Zeitung of 17 May 1800]

[5] Anonymous. (Obituary of Florian Ulbrich). *Annalen der österreichischen Literatur*, 28:217, 1802.

[6] Anonymous. Obras, &. i.e. *the Works of Francis de Borja Stockler* (review). *The Monthly review; or, Literary journal*, 51:490–494, 1806.

[7] Anonymous. (On errors in Felkel's and Vega's tables). *Oesterreichischer Beobachter*, 52:251–252, 1820. [See [58] for a critical answer.]

[8] Anonymous. Ulbrich, Florian. In Johann Jakob Heinrich Czikann and Franz Gräffer, editors, *Oesterreichische National-Encyklopädie, oder alphabetische Darlegung der wissenswürdigsten Eigenthümlichkeiten des österreichischen Kaiserthumes, etc.*, volume 5, page 464. Wien: Friedr. Beck'schen Universitäts-Buchhandlung, 1836.

[9] Raymond Clare Archibald. Felkel's table. *Scripta Mathematica*, 4:336–338, 1936.

[10] Raymond Clare Archibald. New information concerning Isaac Wolfram's life and calculations. *Mathematical Tables and other Aids to Computation*, 4(32):185–200, 1950.

[32]**Note on the titles of the works:** Original titles come with many idiosyncrasies and features (line splitting, size, fonts, etc.) which can often not be reproduced in a list of references. It has therefore seemed pointless to capitalize works according to conventions which not only have no relation with the original work, but also do not restore the title entirely. In the following list of references, most title words (except in German) will therefore be left uncapitalized. The names of the authors have also been homogenized and initials expanded, as much as possible.

The reader should keep in mind that this list is not meant as a facsimile of the original works. The original style information could no doubt have been added as a note, but we have not done it here.

[11] Extrait de la correspondance de M. Bernoulli. *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres, année 1781*, pages 31–35, 1783.

[12] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 1. Berlin, 1782.

[13] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 2. Berlin, 1782.

[14] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 4. Berlin, 1784.

[15] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 5. Berlin: Franz de la Garde, 1785–1787.

[16] Johann Paul Bischoff. *Versuch einer Geschichte der Rechenmaschine, Ansbach 1804*. München: Systhema Verlag, 1990. [edited by Stephan Weiss]

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