Jost Bürgi's skillful computation of sines

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1 Introduction

Folkerts, Launert and Thom have recently announced a very interesting discovery about Jost Bürgi (1552–1632) [3].[1] Bürgi is well known as a (very) skillful mechanician, clockmaker and instrument maker, and also as an inventor of a table of progressions which could be used for the same purpose as logarithms. His table can even be considered a table of logarithms, although we do not consider him as the inventor of logarithms, because there is much more to logarithms than tables [12].

It was also long known that Bürgi had computed a table of sines, but little was known about his methods. His table of sines at intervals of $2''$ does not seem to have survived, and how accurate the table was remains a mystery. In 1588, Ursus gave a rather cryptic hint about Bürgi's method [17] and it seemed to be related to differences. In an earlier article [12], we had theorized that Bürgi had perhaps anticipated later work where pivot values were computed, then differenced, and then the differences used to compute intermediate values in a manner similar to that used in the *Tables du Cadastre* [13].

Bürgi's newly discovered manuscript on trigonometry gives a number of answers to four centuries of questions. But it also raises new questions.

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[1] We thank Fritz Staudacher for having informed us of that recent discovery on October 23, 2015.

2 Bürgi's algorithm

2.1 Dividing the quadrant in nine parts

Folkerts *et al.* [3] give the following table adapted from Bürgi's manuscript dated from the 1580s. Bürgi used instead sexagesimal values, but we can safely convert them to base 10.

	c_5		c_4		c_3		c_2		c_1
0	0		0		0		0		0
		2,235,060		67,912		2,064		63	
10	2,235,060		67,912		2,064		63		2
		2,167,148		67,848		2,001		61	
20	4,402,208		133,760		4,065		124		4
		2,033,388		61,783		1,877		57	
30	6,435,596		195,543		5,942		181		6
		1,837,845		55,841		1,696		51	
40	8,273,441		251,384		7,638		232		7
		1,586,461		48,203		1,464		44	
50	9,859,902		299,587		9,102		276		8
		1,286,874		39,101		1,188		36	
60	11,146,776		338,688		10,290		312		9
		948,186		28,811		876		27	
70	12,094,962		367,499		11,166		339		10
		580,687		17,645		537		17	
80	12,675,649		385,144		11,703		356		11
		195,543		5,942		181		6	
90	12,871,192		391,086		11,884		362		12

The purpose of this table is to compute the values of $\sin 10°$, $\sin 20°$, ..., $\sin 90°$, to any desired accuracy. Bürgi's algorithm is deceptively simple. There is no bisection, no roots, only two basic operations: many additions, and a few divisions by 2. The computations can be done with integers, or with sexagesimal values, or in any other base.

In order to compute the sines, Bürgi starts with an arbitrary list of values, which can be considered as first approximations of the sines, but need not be. These values are given in column c_1. In all these columns, c_1 to c_5, the last value is always the *sinus totus*. That is, the first approximation starts with $\sin 90° = 12$. This gives in modern terms for $\sin 60°$ the value $\frac{9}{12} = 0.75$, an approximation of the actual sine which is $0.866\ldots$. It basically matters very little with what values one starts. Only the first value should be 0. It is even possible to take all other initial values equal to 1, for instance, but they can't all be taken equal to 0.

Column c_5 shows the result of that algorithm, here conducted to four steps, but the table could have been extended at will towards the left. Now, in column c_5, there is a new value for the *sinus totus*, namely 12871192, and therefore we have as a new approximation of $\sin 60°$ the fraction $\frac{11146776}{12871192} = 0.86602515136$, the exact value being $\frac{\sqrt{3}}{2} = 0.866025403\ldots$.

Bürgi's algorithm is an iterative procedure for computing all the values of column c_{i+1} from those of column c_i. The computations use an intermediate column whose last value is half of the previous *sinus totus*. In the above example, the *sinus totus* in column c_1 is 12, and the last value of the column between c_2 and c_1 is 6. The last value of the column between c_3 and c_2 is 181, half of 362. If the *sinus totus* is odd, one might take the exact half, but it actually does not matter, as this algorithm leads to increasingly larger numbers, and ignoring a half integer only has marginal consequences on the convergence.

Once the last value of an intermediate column has been obtained, all other values of that column are obtained by adding the values in the previous column, as if the previous column were differences. So, we have $6 + 11 = 17$, $17 + 10 = 27$, and so on.

When the intermediate column has been filled, the new column c_{i+1} is constructed by starting with 0, and adding the values of the intermediate column.

And that's all!

2.2 Other subdivisions

The previous section led to the computation of the sines of 10°, 20°, ..., 90°, without any geometric construction. This scheme can be adapted to obtain any set of values $\sin(\frac{k\pi}{2n})$, for $0 \le k \le n$, but it is important to realize that this algorithm provides all the values, and cannot provide only one sine. All the sines are obtained in parallel, with relatively simple operations.

The value of n corresponds to the number of values in the intermediate columns. In the above example, $n = 9$. If we start with $n = 90$, we will obtain the sines of all degrees in the quadrant.

We can also conduct simpler examples, such as computing the sine of 45° by taking $n = 2$. If we start with the three values 0,1,2 (say), that is, with 0.5 for the approximation of $\sin 45°$, we obtain the intermediate values 2,1, from which we get the new approximations 0,2,3 and hence $\frac{2}{3} = 0.666\ldots$ for the sine. Continuing this process, we obtain the intermediate values 3,1 (or 3.5,1.5 if we do not round), leading to the new approximations 0,3,4, and $\frac{3}{4} = 0.75$ for the sine. Still going on, we find the intermediate values 5,2 and the new approximations 0,5,7, hence $\frac{5}{7} = 0.71428\ldots$ for the sine. This procedure will converge towards the value $\frac{\sqrt{2}}{2} = 0.707106781\ldots$.

3 Bürgi's intuition

How did Bürgi obtain this algorithm? Folkerts *et al.* [3] do not answer this question. They give a modern proof of the convergence. This proof, by Andreas Thom, is quite intricate and is based on the observation that the linear map between one set of approximations and the next one has eigenvalues which are the sought sines.

Bürgi did not have these means, and at first we thought that Bürgi's algorithm must be the expression of a clever geometrical construction. After some search, we gave up this idea.

Then, we found a very simple way to derive Bürgi's algorithm. As mentioned above, Bürgi's procedure alludes to differencing, and we know that Bürgi did use differences to compute other sines between $\sin 1'$ and $\sin 1°$. Now, in the tables of Rheticus and Viète, trigonometrical values were given with first differences, in order to be able to check these values.

Neither Rheticus in 1551 [7], nor Viète in 1579 [18] gave second differences. Rheticus's calculations for the *Opus palatinum* [8] may have contained second differences, but they do not appear in the table published in 1596. Second differences only appeared in Pitiscus's *Thesaurus mathematicus* in 1613 [6].

Bürgi must have been looking with great care at Rheticus' values, and toyed with the computation of differences, including second differences. He must then have noticed that the second differences of the sines mimic the sines themselves. In other words, the second differences are approximations to the sines, and the more one differences, the worse these approximations are. Conversely, if there was a way to start with approximate differences and go backwards towards the actual sines, accuracy would be obtained. But was this feasible?

In order to make it work, or at least to see if one could go backwards, only two things were needed:

- Bürgi had to notice (without proof) that there was almost a constant ratio between the sines and there second differences; in the example for the computation of $\sin 10°$, $\sin 20°$, that ratio is about 32.911 (in fact $\frac{\csc^2(\pi/36)}{4}$ as highlighted by Thom); this ratio depended on the number of subdivisions, but what was important was that the second differences were close to being sines; when second differences are obtained from the sines, there are two values less than at the beginning; the first value could easily be supplied, it had to be 0; the last value could also be supplied, using the known ratio between sines and second differences;

- The second important observation was to notice that the last value of the first differences is very nearly half the value added at the end of the second differences.

Once these two observations were made, and once they were checked to be true whatever the subdivision, Bürgi must have been trying to go backwards, that is, to apply to algorithm given above. He may have started with approximate values of the sines, and perhaps didn't notice that the algorithm converges even with practically random initial values.

We therefore believe that Bürgi was able to discover some properties of finite differences by a careful observation and manipulation of actual differences, and that he tried to reverse the procedure, and must have been pleased to see that it did in fact make it possible to compute sines with only simple operations, and at any desired accuracy. It seems unlikely to us that Bürgi had a proof, or that he sought one.[2] It is therefore certainly a very skillful and remarkable achievement.

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[2]Folkerts *et al.* [3] mention that Peter Ullrich has also proposed a conjecture on how Bürgi may have found his method, but we do not know if Ullrich's conjecture is identical or not with ours.

quadrantis : una cum sinibus primi et postremi gradus, ad eundem radium, et ad singula scrupula secunda quadrantis : adiunctis ubique differentiis primis et secundis; atque, ubi res tulit, etiam tertijs. Frankfurt: Nicolaus Hoffmann, 1613. [The tables were reconstructed by D. Roegel in 2010. [9]].

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