Napier's bones
and Genaille-Lucas's rods

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Abstract

This note presents a survey of Napier's bones and of their more "automatic" evolution, the Genaille-Lucas rods. Generalizations of both to other bases are also shown.

Contents

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1 Napier's bones

1.1 Napier's original bones

In his "Rabdology" published in 1617 under the title *Rabdologiæ, seu numerationis per virgulas libri duo* [26], John Napier (1550–1617) introduced his famous "bones." Three years after the seminal work on logarithms [25, 33], here was a much more practical tool for everyday calculations.

Napier's bones were aimed at facilitating multiplications and divisions. A basic set of "bones" is made of 11 rods, one for each integer from 0 to 9, and an additional one serving as an index. More rods can be used, and several basic sets can be combined at will.

Figure 1 shows the basic set of bones. Each bone from 0 to 9 carries the multiples of the corresponding value. Units and tens are separated by a diagonal. For instance, the multiples of 7 are 7, 1/4, 2/1, 2/8, 3/5, etc. The diagonals will be used to transfer carries naturally. The rods actually have four faces, so that each rod actually carries four sets of multiples, often making it unnecessary to use rods from another set to complement the basic ones.

Figure 2 shows how these rods are used for a multiplication. If we wish to compute 4×1897, we take the rods 1, 8, 9, and 7, and we put them next to each other, at the right of the special index rod. On line 4 (marked in yellow), we can read the result. The digits can be read from the right and each diagonal strip must be evaluated to one digit. In that example, the rightmost digit is therefore 8, then comes $2+6=8$ (where we intentionally used the same color), then $3+2=5$, then $3+4=7$, and the result is therefore 7588. In general, there will be carries from one strip to the next, but not in this simple example.

Figure 3 shows a more complex multiplication, involving several identical rods, as well as carries. Upper left blank triangles correspond to the value 0, and we represent them as Napier did.

In order to multiply a number by another one greater than 9, one computes the partial products with the bones, and the addition of these products is performed by hand.

Napier's bones can also be used for division. The same rods are used for multiplying a number and for dividing by that number. So, if we want to divide a number by 4257, we will use the four rods 4, 2, 5, and 7 (figure 4). Let's for instance divide 123456 by 4257. We start with the first five digits of the number, 12345, as this is the smallest prefix greater than the divider. And using the bones, we seek the greatest multiple of 4257 not larger than 12345. It is the second multiple, 8514. We then subtract 8514 to 12345, yielding 3831, and since we have used the second multiple, we know that the first digit of the quotient is 2. Then, we add the next digit from the initial number, yielding 38316. To this we subtract the largest possible multiple of 4257, namely the ninth multiple, 38313, and the remainder is 3. Consequently, we have found that $123456 = 29 \times 4257 + 3$. Even though this looks complex to our 21st century's eyes, we have to realize that the bones made it essentially easy to provide the multiples not only of 0 to 9, but of any number, with the need of only a few additions of carries here and there. The human operator did not need to know much more than very limited additions.

Napier also provided two additional rods for their use in the extraction of square and cubic roots which are not described here. Finally, he devised a way to facilitate the multiplication of multi-digit numbers. His tool, the *promptuarium*, prealigns the various partial products, so that one then merely has to add digits in diagonals, not only for one partial product, but for all of them at the same time. We will also not describe these extensions here, but perhaps in another article.

Napier's bones were used in several "machines", in particular in Schickard's calculating machine (1623), as an aid for the computer. The carries were never performed automatically. A number of constructions, such as those of Schott (1668) and Leupold (1727), have made it easier to manipulate the rods, but without substantially improving the original design.

The main problem of Napier's bones is of course that the multiples are not given in their final form, and there remains the need to perform additional additions and carries. Performing the additions can be made (presumably) slightly easier if the figures to be added are laid out vertically, and "slanted" rods for that purpose have been manufactured in the 19th century. Another partial solution was that of Raussain (1738) who used various colors in order to facilitate the alignments, but it was still necessary to add the carries [19]. An effective and practical solution, however, was only made available around 1884 by Genaille and Lucas.

Napier's bones, as well as the solution of Genaille and Lucas, are still occasionally used nowadays as educational tools [34].

I	0	1	2	3	4	5	6	7	8	9
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	1/0	1/2	1/4	1/6	1/8
3	0	3	6	9	1/2	1/5	1/8	2/1	2/4	2/7
4	0	4	8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
5	0	5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
6	0	6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
7	0	7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
8	0	8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
9	0	9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

Figure 1: The Napier bones. There is one rod for every integer from 0 to 9, and an index rod to the left.

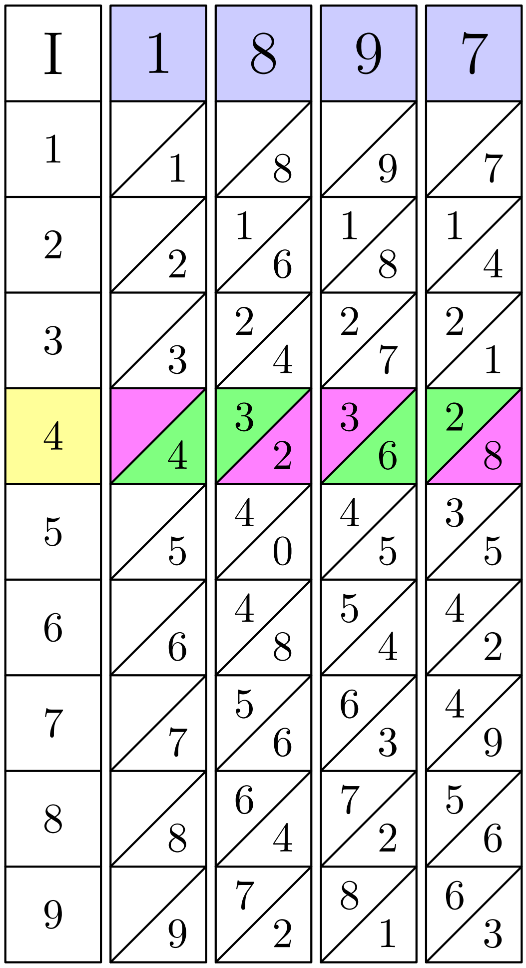

Figure 2: A calculation example: $4 \times 1897 = /4 + 3/2 + 3/6 + 2/8/ = /7/5/8/8/ = 7588$.

I	3	8	7	7	0	1	9
1	/3	/8	/7	/7	/0	/1	/9
2	/6	1/6	1/4	1/4	/0	/2	1/8
3	/9	2/4	2/1	2/1	/0	/3	2/7
4	1/2	3/2	2/8	2/8	/0	/4	3/6
5	1/5	4/0	3/5	3/5	/0	/5	4/5
6	1/8	4/8	4/2	4/2	/0	/6	5/4
7	2/1	5/6	4/9	4/9	/0	/7	6/3
8	2/4	6/4	5/6	5/6	/0	/8	7/2
9	2/7	7/2	6/3	6/3	/0	/9	8/1

Figure 3: Another calculation example: $7 \times 3877019 = /2/1 + 5/6 + 4/9 + 4/9/0/7 + 6/3/ = /2/6/10/13/9/0/13/3/ = /2/7/1/3/9/1/3/3/ = 27139133$.

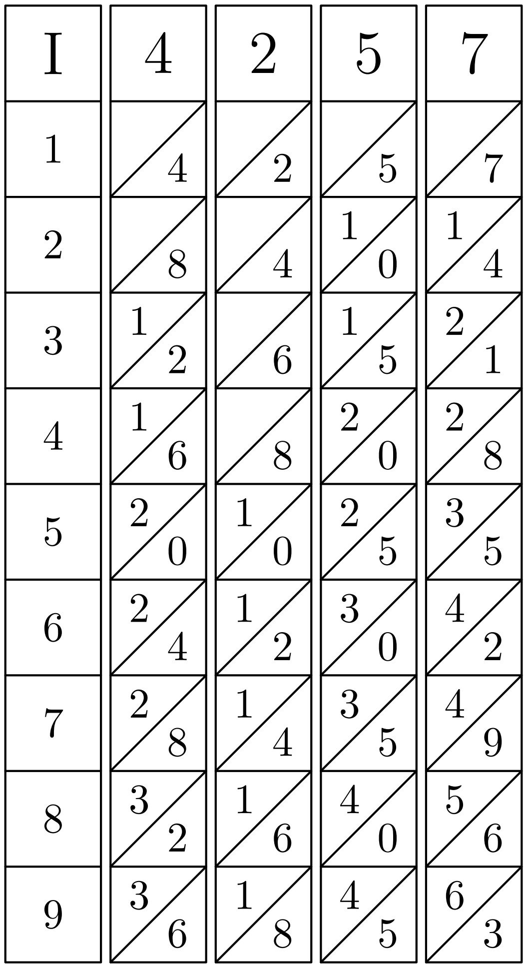

Figure 4: When dividing by 4257, the rods 4, 2, 5, and 7 are used, and then they provide the various multiples needed to perform the division (see main text).

1.2 Napier's bones in other bases

Napier's bones can be extended to other bases. Figures 5, 6, 7, 8, 9, and 10 show the bones for bases 2, 3, 5, 8, 11, and 16. Figures 11 and 12 show computations in base 7 and 16.

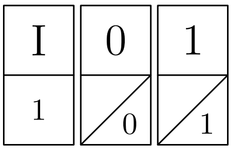

Figure 5: Napier's bones in base 2.

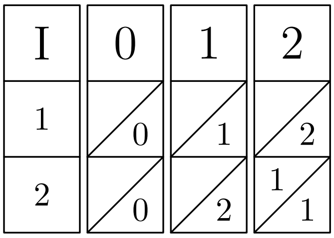

Figure 6: Napier's bones in base 3.

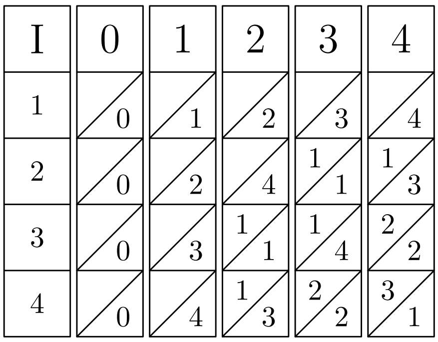

Figure 7: Napier's bones in base 5.

I	0	1	2	3	4	5	6	7
1	0	1	2	3	4	5	6	7
2	0	2	4	6	1/0	1/2	1/4	1/6
3	0	3	6	1/1	1/4	1/7	2/2	2/5
4	0	4	1/0	1/4	2/0	2/4	3/0	3/4
5	0	5	1/2	1/7	2/4	3/1	3/6	4/3
6	0	6	1/4	2/2	3/0	3/6	4/4	5/2
7	0	7	1/6	2/5	3/4	4/3	5/2	6/1

Figure 8: Napier's bones in base 8 (octal).

I	0	1	2	3	4	5	6	7	8	9	A
1	0	1	2	3	4	5	6	7	8	9	A
2	0	2	4	6	8	A	1/1	1/3	1/5	1/7	1/9
3	0	3	6	9	1/1	1/4	1/7	1/A	2/2	2/5	2/8
4	0	4	8	1/1	1/5	1/9	2/2	2/6	2/A	3/3	3/7
5	0	5	A	1/4	1/9	2/3	2/8	3/2	3/7	4/1	4/6
6	0	6	1/1	1/7	2/2	2/8	3/3	3/9	4/4	4/A	5/5
7	0	7	1/3	1/A	2/6	3/2	3/9	4/5	5/1	5/8	6/4
8	0	8	1/5	2/2	2/A	3/7	4/4	5/1	5/9	6/6	7/3
9	0	9	1/7	2/5	3/3	4/1	4/A	5/8	6/6	7/4	8/2
A	0	A	1/9	2/8	3/7	4/6	5/5	6/4	7/3	8/2	9/1

Figure 9: Napier's bones in base 11.

I	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
2	0	2	4	6	8	A	C	E	1/0	1/2	1/4	1/6	1/8	1/A	1/C	1/E
3	0	3	6	9	C	F	1/2	1/5	1/8	1/B	1/E	2/1	2/4	2/7	2/A	2/D
4	0	4	8	C	1/0	1/4	1/8	1/C	2/0	2/4	2/8	2/C	3/0	3/4	3/8	3/C
5	0	5	A	F	1/4	1/9	1/E	2/3	2/8	2/D	3/2	3/7	3/C	4/1	4/6	4/B
6	0	6	C	1/2	1/8	1/E	2/4	2/A	3/0	3/6	3/C	4/2	4/8	4/E	5/4	5/A
7	0	7	E	1/5	1/C	2/3	2/A	3/1	3/8	3/F	4/6	4/D	5/4	5/B	6/2	6/9
8	0	8	1/0	1/8	2/0	2/8	3/0	3/8	4/0	4/8	5/0	5/8	6/0	6/8	7/0	7/8
9	0	9	1/2	1/B	2/4	2/D	3/6	3/F	4/8	5/1	5/A	6/3	6/C	7/5	7/E	8/7
A	0	A	1/4	1/E	2/8	3/2	3/C	4/6	5/0	5/A	6/4	6/E	7/8	8/2	8/C	9/6
B	0	B	1/6	2/1	2/C	3/7	4/2	4/D	5/8	6/3	6/E	7/9	8/4	8/F	9/A	A/5
C	0	C	1/8	2/4	3/0	3/C	4/8	5/4	6/0	6/C	7/8	8/4	9/0	9/C	A/8	B/4
D	0	D	1/A	2/7	3/4	4/1	4/E	5/B	6/8	7/5	8/2	8/F	9/C	A/9	B/6	C/3
E	0	E	1/C	2/A	3/8	4/6	5/4	6/2	7/0	7/E	8/C	9/A	A/8	B/6	C/4	D/2
F	0	F	1/E	2/D	3/C	4/B	5/A	6/9	7/8	8/7	9/6	A/5	B/4	C/3	D/2	E/1

Figure 10: Napier's bones in base 16 (hexadecimal).

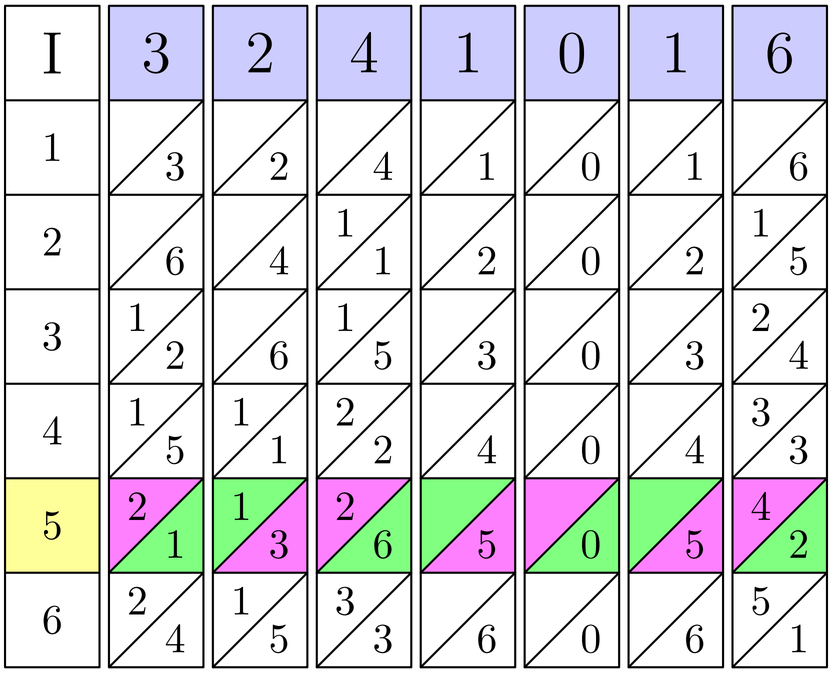

Figure 11: A computation in base 7.

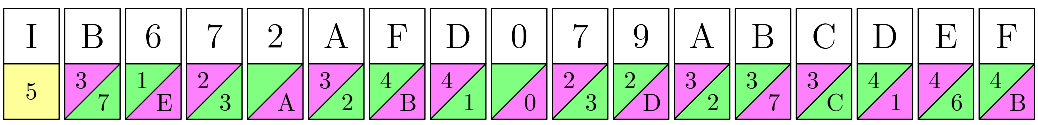

Figure 12: A computation in base 16 (only the 5th multiple is shown).

2 The Genaille-Lucas rods

Henri Genaille (18??–1903) was an engineer of the French railways.[1] He invented a number of arithmetic constructions and several of them are kept in the collections of the CNAM in Paris [1]. At the end of the 1880s, he constructed for instance a *piano arithmétique*, a machine for finding the primes of the form $2^n - 1$ [41, pp. 493–494].

Around 1878 [6], he came up with a new computation method which was presented at the 7th meeting of the *Association française pour l'avancement des sciences* in 1878 in Paris. At that time, Genaille apparently did not yet have the idea of using rods, but the underlying concepts were already there. In particular, one commentator, Laisant[2], observed that the "lines" drawn on Genaille's board could be replaced by grooves and that a small ball could be led through it, and its path could then automatically let appear the result of a computation, and in a certain way, the computation would be performed by gravity.

Genaille's method was then probably improved and his new rods were mentioned by the French mathematician Édouard Lucas (1842–1891) [2, 3] at the 1884 meeting of the *Association française pour l'avancement des sciences* in Blois [21]. In 1885, Genaille and Lucas then published four sets of calculating rods.[3] The first set was a set of Napier bones [12]. The three other sets were Genaille's inventions and were aimed at simplifying multiplications and divisions [10, 11, 9].[4] They were highlighted as being "exact" and "instantaneous." According to a review published in 1885 [5], the boxes of rods were prepared for the 1889 universal exhibition in Paris. Lucas briefly mentioned these rods in his posthumous third volume of *Récréations mathématiques* [22], where he stated that the idea of the rods was Genaille's, but that Lucas added the three other faces.

The main motivation of the Genaille-Lucas rods is to dispense of the carries in Napier's bones. Genaille and Lucas basically made two sets of rods, one for the multiplication, and another for the division, in which both operations can be done smoothlessly. The operator never performs any addition and is not aware of any carries. This is possible because the manipulation of carries is made so discrete,

[1]In the 1883 list of members of the *Association française pour l'avancement des sciences*, he is listed as "Ingénieur civil au bureau central des chemins de fer de l'État," and his address was "16 rue Saint-Étienne" in Tours. In 1878, he was engineer in Lyon. Of Genaille's life we know very little, except that he died on May 16, 1903 [30, p. 14].

[2]Probably Charles-Ange Laisant (1841–1920).

[3]The exact chronology of the publication of these rods is often given incorrectly. Some authors, for instance Williams [42, 43, 44], write that Lucas presented a problem on arithmetic at the 1885 meeting of the *Association française pour l'avancement des sciences*, when in fact he already mentioned the existence of Genaille's rods at the previous meeting in 1884. The presentation of the problem must therefore have occurred earlier, perhaps already in 1878. Boxed sets of the rods were published in 1885, but they have possibly been displayed at the 1884 meeting.

[4]For the original instructions which came with these rods, see Weiss's articles [38, 39].

so natural, that one feels that there are no carries at all! The division set also had a more specialized form for financial calculations.

2.1 Multiplication

2.1.1 The original multiplication rods

In order to (apparently) dispense of the carries, Genaille and Lucas have actually packed more information into each cell. Consider for instance Napier's bone 4, and its seventh multiple. It contains the two figures 2 (tens) and 8 (units) (see figure 13). But if this is part of a wider product, there may be figures anywhere between 0 and 6 coming in and added to the units. In other words, the actual units may be 8, but also 9, 0, 1, 2, 3, and 4. In the last four cases, the tens will no longer be 2, but 3. All these cases are possible and are made explicit in a cell of the Genaille-Lucas rods. Figure 14 shows three cells, and how they encode the various possibilities. A cell contains a numerical representation of the values of its possible units (for instance 8, 9, 0, 1, 2, 3, and 4), and a graphical representation of the possible values of the tens. For each value of the units, there is a corresponding value of the tens. The first cell in figure 14 shows that if there is no incoming carry (0), the units will be 8, and the tens will be 2 ($4 \times 7 = 28$). The second cell is identical to the first, but shows what happens if the incoming carry is 2, the units will then be 0, and the tens will be 3 ($4 \times 7 + 2 = 30$). The third cell illustrates that there is a relationship between cells (n,m) and (m,n) in that one is a extension (or truncation) of the other. The third cell is here shorter, because there are only four possible incoming carries.

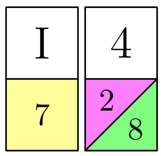

Figure 13: The information contained in one cell of Napier's bones.

Each cell contains one or two triangles. It can never contain more than two triangles, because the transition from the first triangle to the second one corresponds to the passage of the units from 9 to 0, and there can be only one such transition in a cell.[5]

[5]The main features of Genaille's rods for multiplication were however already given by Karl Schönbichler in 1850, see the account by Weiss [38]. But Schönbichler apparently didn't go so far as to market the new rods.

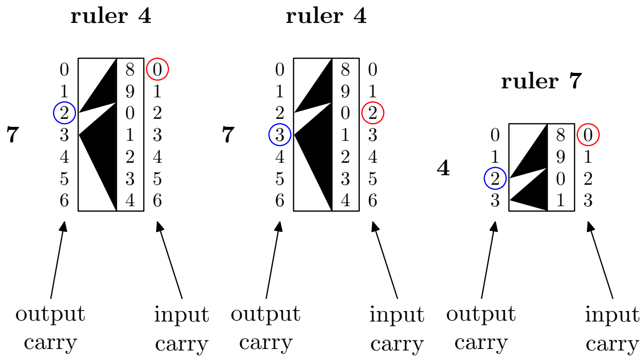

Figure 14: A cell (left and middle) and its conjunct (right).

Figure 15 shows the entire set of Genaille-Lucas rods for multiplication. There are ten rods giving all possibilities when multiplying the rod's value by 2, 3, upto 9. Genaille and Lucas started with 2, as 1 is not really needed.

Figures 16 and 17 show how the rods are actually used and how effective the calculation is. In order to compute 3×567439, we take the rods 5, 6, 7, 4, 3, and 9, as we would have done with Napier's bones. Then, on the row marked 3 on the left (index) rod, we start on the first line at the right, here marked with a black wedge. This gives the first digit, 7. We continue from right to left, merely be following the triangles. We always stay in the same triangle, and move from the base (right) to the tip (left). So, 7 leads to 1, which leads to 3, which leads to 2, which leads to 0, which leads to 7, which leads to 1. And we are done! The result is 1702317. No apparent carries. Of course, the truth is that when moving from the first to second rod, we have added 2, by taking the last entry of the second cell, and similarly with the other transitions, but all this took place very naturally. We did not actually perform additions, we took the paths corresponding to the required additions. It is possible to consider improvements to these rods, one of them being to hide the figures of a rod which will never be used. This can be achieved if the rods are allowed to overlap and carry holes, as illustrated by Weiss [38]. It is also possible to write all the rods on cylinders and to show only a portion of each cylinder, thereby basically providing ten rods at each place. Such a "calculating machine" has been rebuilt by Valéry Monnier in 2011. Several variant constructions are possible.

For multiplying by numbers containing several digits, another construction has been proposed [21], [22, p. 82], but it is still cumbersome, and Genaille and Lucas had planned to construct an electrical calculating machine (in 1884!) for solving these problems [7], a machine that unfortunately never came into being.[6]

[6]A revival of this early idea was the machine patented by Nicoladze in 1928 and which was basically an electrical wiring of Genaille's triangles [29, 28].

The machine was to compute the product of two numbers of 10 digits, and was going to cost less than 20 francs [22, p. 83]. In 1894, Genaille presented another machine for general multiplications, but it is not based on the Genaille rods [8].

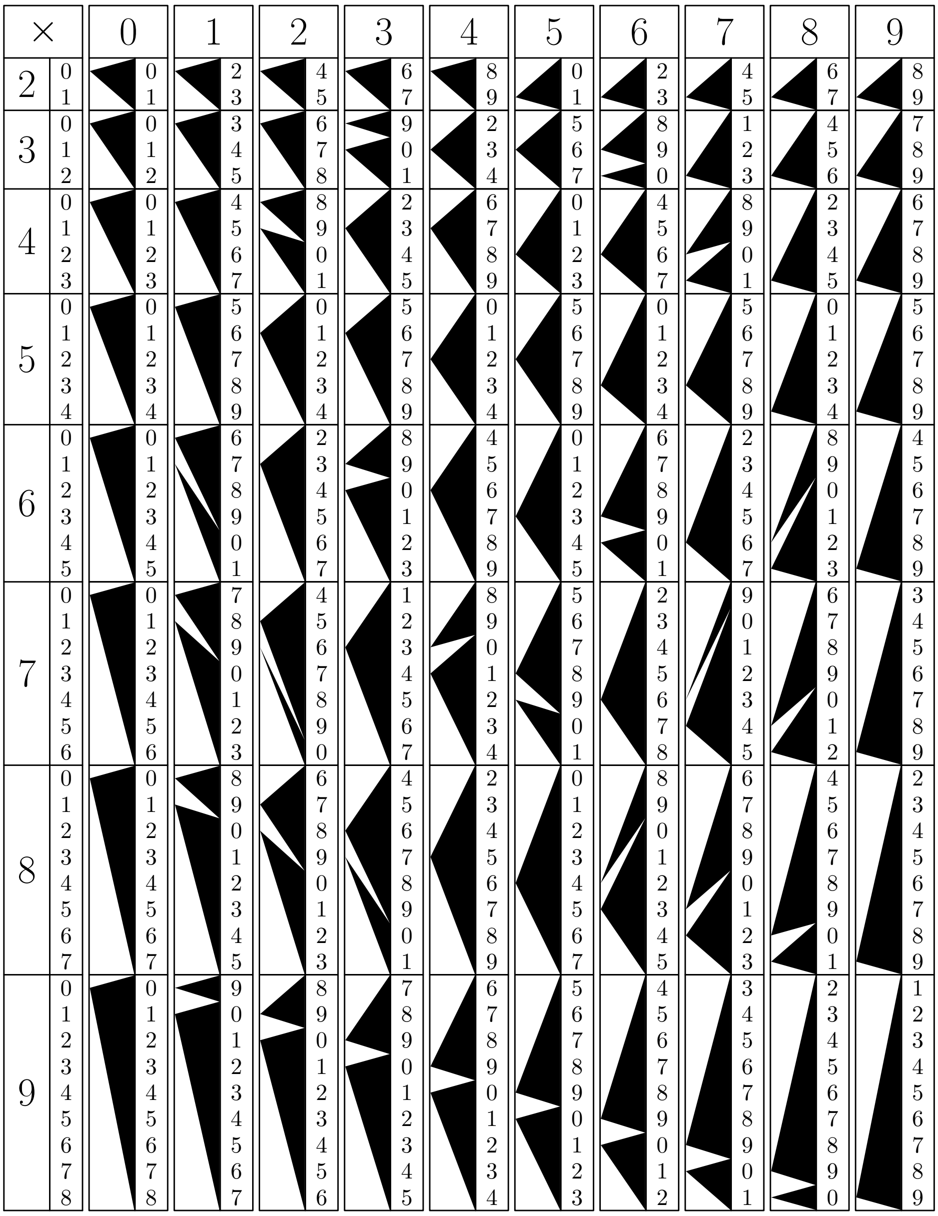

Figure 15: The Genaille-Lucas rods.

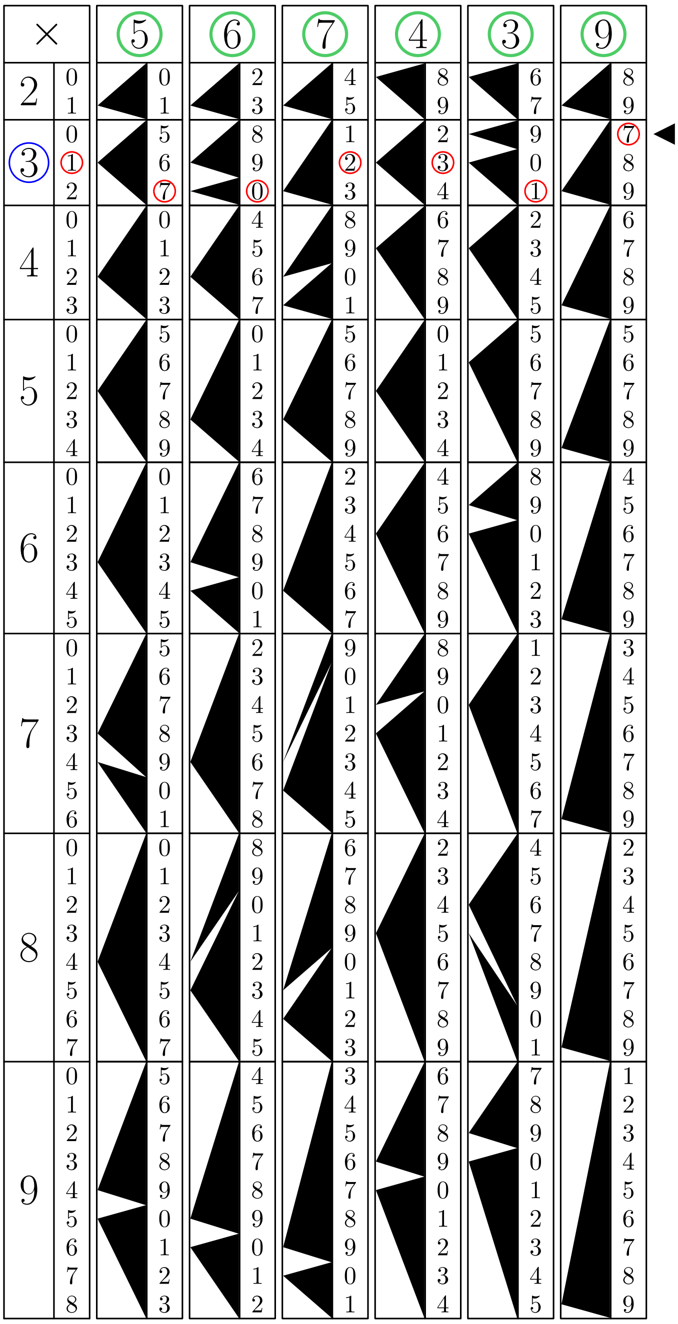

Figure 16: A computation example.

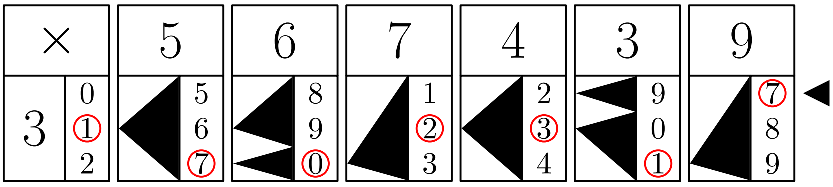

Figure 17: The same example, with only the relevant row.

2.1.2 Multiplication rods in other bases

The Genaille-Lucas multiplying rods can easily be extended to other bases, and their properties remain the same. In particular, there are still at most two triangles per cell, but the graphical representation of triangles obviously becomes inadequate for higher bases, as it may be difficult to separate the triangles. Figure 18 shows the Genaille-Lucas rods in base 6, figure 19 shows them in base 16, where they really become difficult to use. Figure 20 shows the details of two base-16 (hexadecimal) cells, the second one with a very thin triangle. We could of course adopt a different representation, but the pure Genaille-Lucas representation obviously reaches its limits. Figures 21 and 22 illustrate two multiplications in base 16.

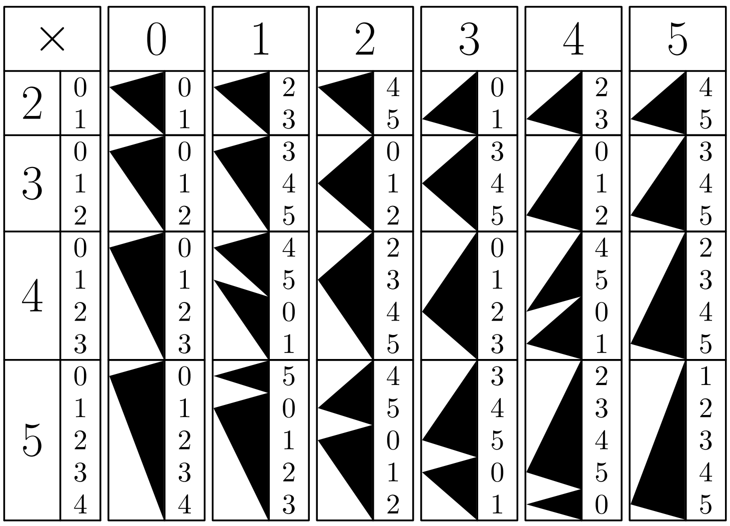

Figure 18: The Genaille-Lucas rods in base 6.

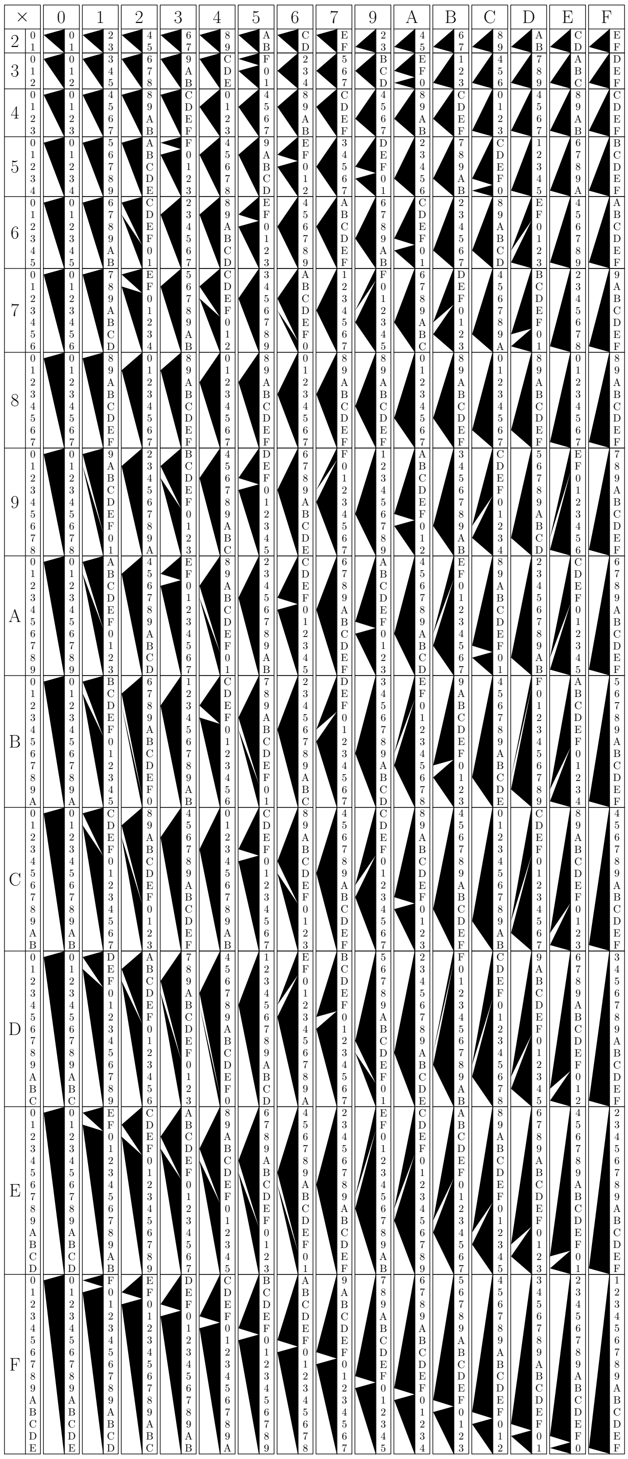

Figure 19: The Genaille-Lucas rods in base 16.

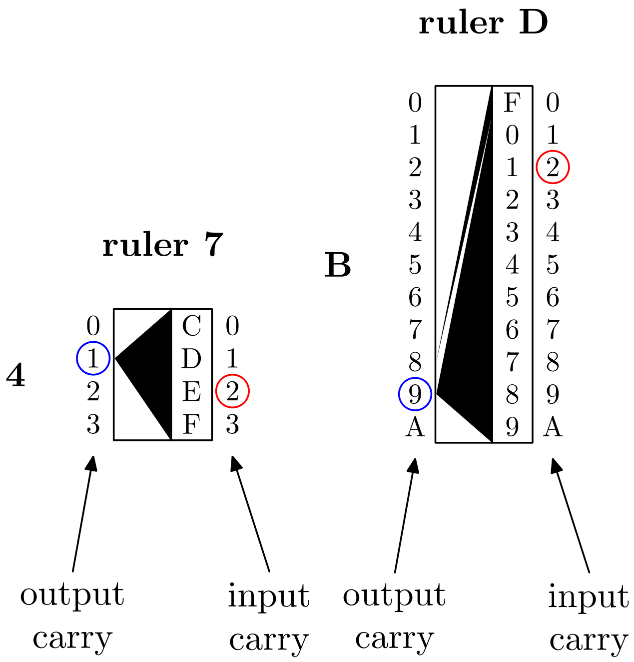

Figure 20: Two cells in base 16.

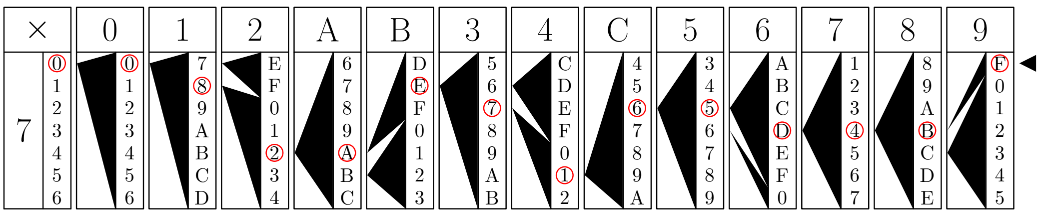

Figure 21: Multiplying 012AB34C56789 by 7 in base 16.

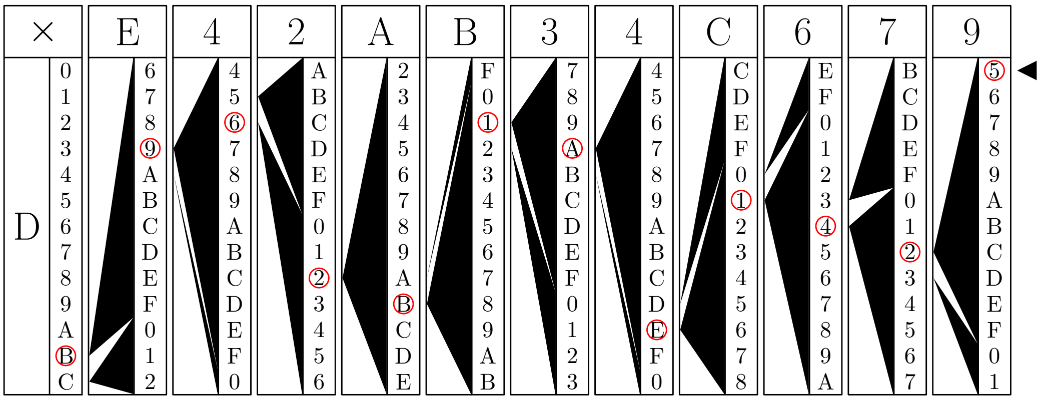

Figure 22: Multiplying E42AB34C679 by D in base 16.

2.2 Division

Napier's bones made it possible to divide numbers, and the rods were basically used to provide multiples of the divider, so that the operator didn't have to memorize them. However, it was still necessary to find the largest multiple, to fulfill its calculation (add the carries), and to subtract it from a prefix of the number to be divided.

2.2.1 The original division rods

Genaille and Lucas devised a new set of rods which did not automate all of these tasks, but provided a simple solution in a special case, namely that of dividing by a one-digit number. Contrary to Napier's bones where the bones chosen corresponded to the divider, with Genaille-Lucas's rods, one selects the rods corresponding to the number to be divided. An index then provides an entry to the (one-digit) divider.

Cells now contain all the possible quotients and remainders. For instance, figure 23 shows the cell for dividing ...7 by 6. Since the previous remainder is smaller than 6, this cell encodes the divisions of 07 by 6, 17 by 6, 27 by 6, etc., until 57 by 6. These six different divisions are shown by the six entry points at the left of the cell. And for each division, the line leads to the corresponding remainder. In the example shown, dividing 17 by 6 results in a quotient of 2 (in blue) and a remainder of 5 (in red). The remainder will become the entry point of the cell located at its right. All cells are encoded similarly.

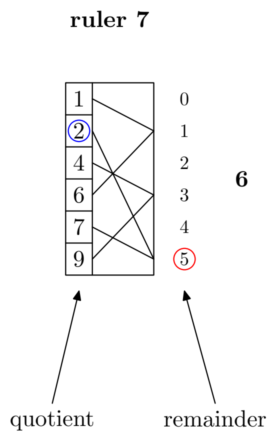

Figure 23: The structure of a cell.

Figures 24 and 25 show computations examples and figure 26 shows the entire set of rods for division.

In figure 24, we divide 536027 by 4, and the entry point is the first line of the leftmost cell corresponding to the divider 4. It is the first line, because the incoming (left) remainder is naturally 0. Then, one has merely to follow the lines, and the result of the division can readily be read as 134006, with a remainder of 3: $536027 = 4 \times 134006 + 3$. Additionnal figures can be obtained by adding additional 0 rods.

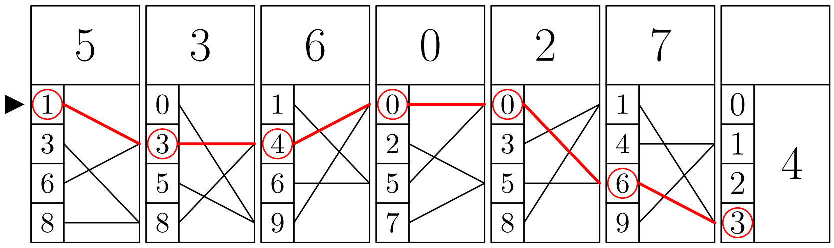

Figure 24: A computation example.

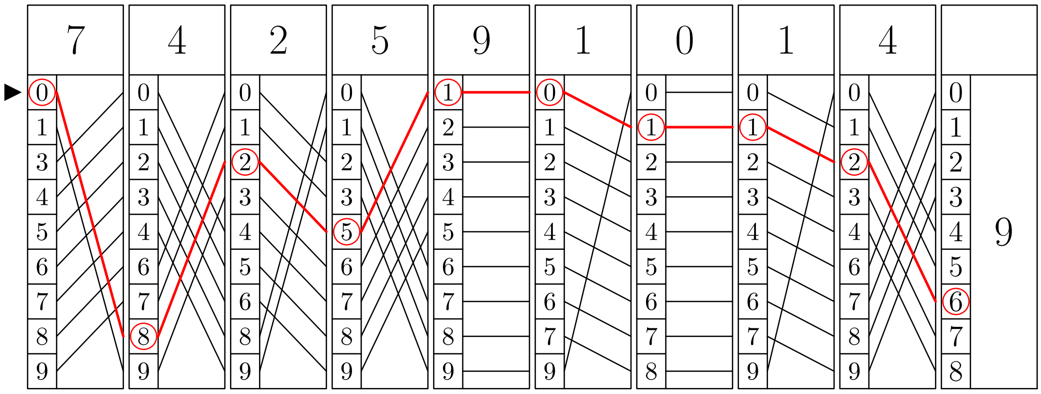

Figure 25: Another computation example.

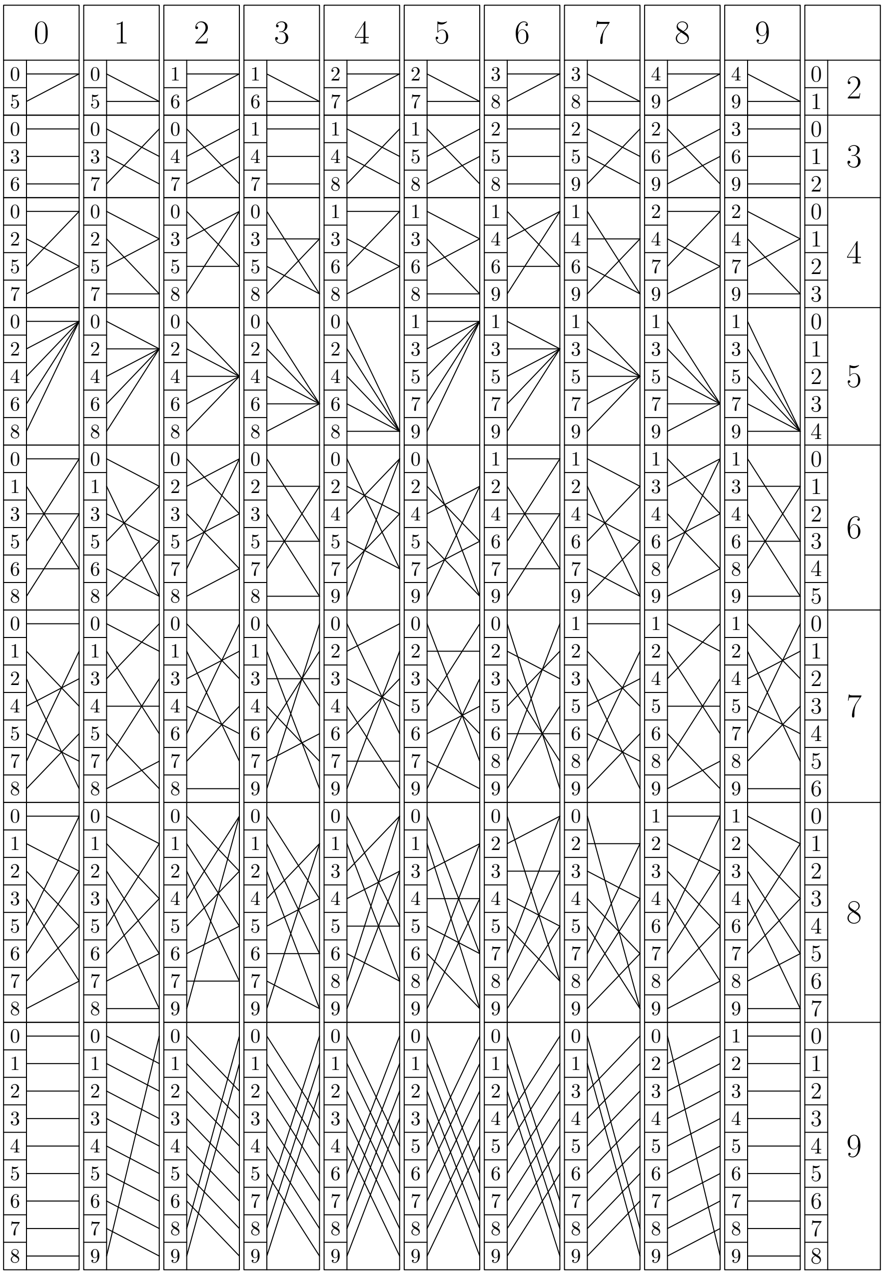

Figure 26: The complete set of rods for division.

2.2.2 Division rods in other bases

Likewise, the division rods can be extended to other bases, and figure 27 shows the rods for base 8. Figure 28 shows three examples using the base 16 rods, but they are admittedly not that practical. Figure 29 show the structure of a hexadecimal (base 16) cell, and figure 30 shows a hexadecimal division restricted to one row of the rods.

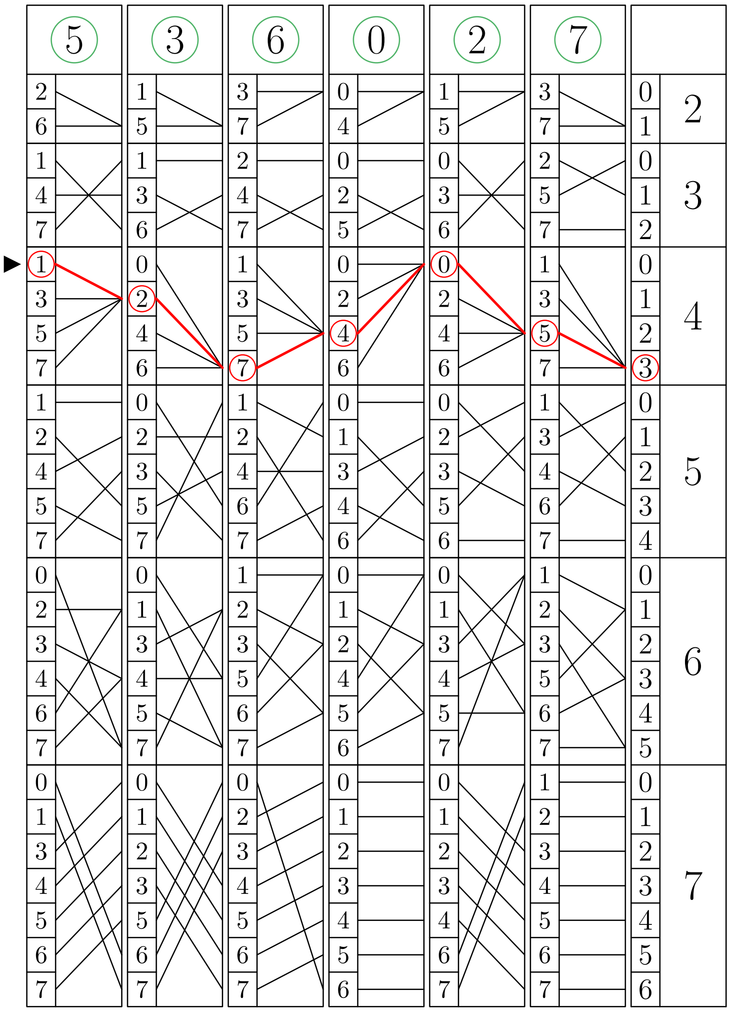

Figure 27: The division rods for base 8.

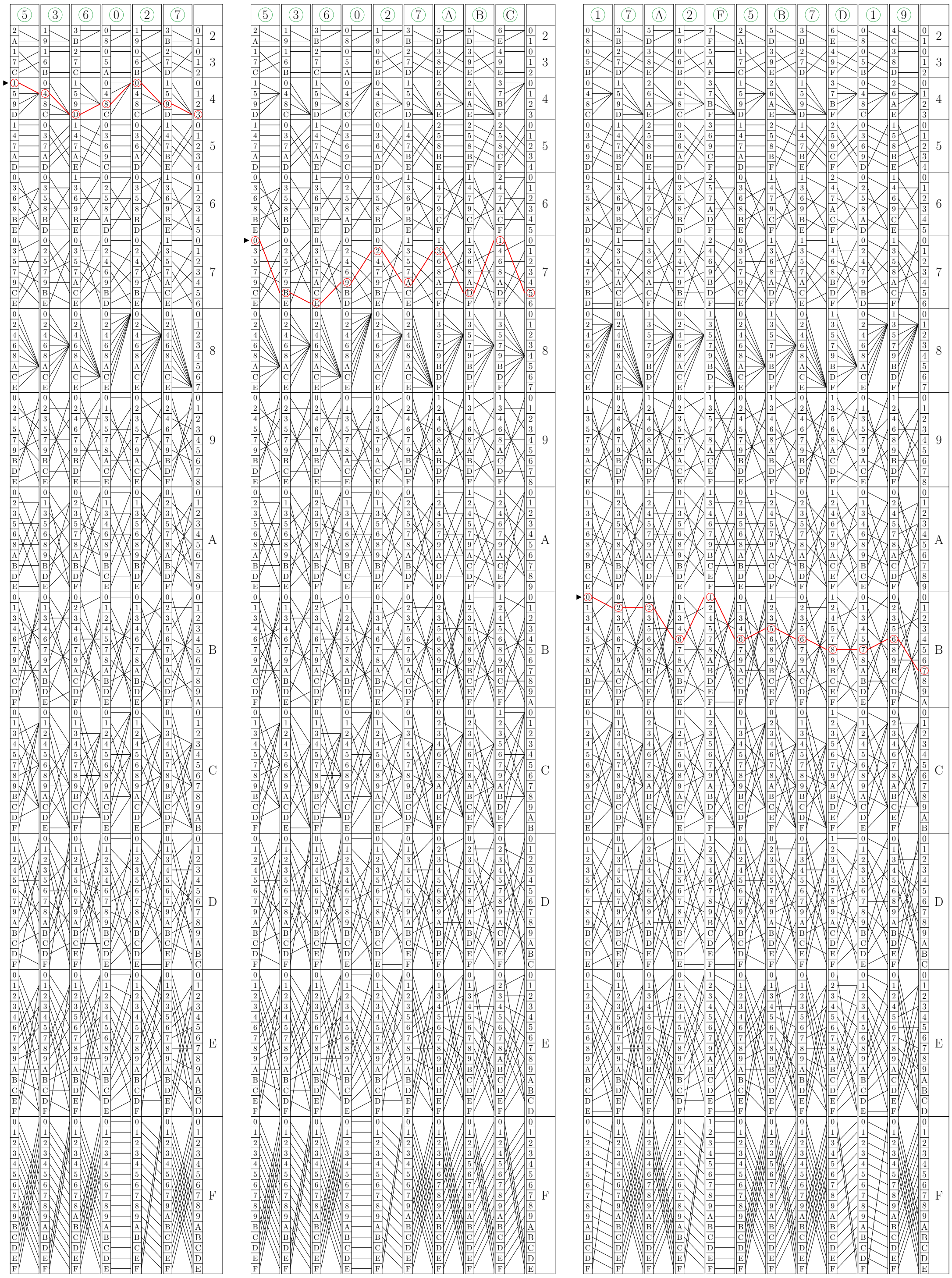

Figure 28: Three examples in base 16.

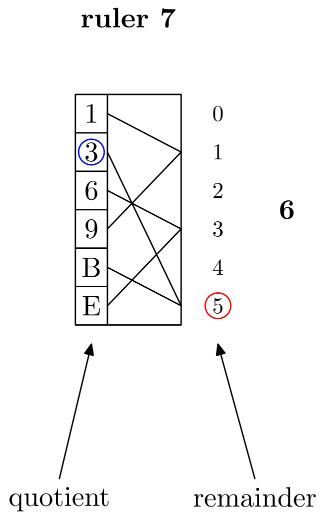

Figure 29: A division cell in base 16.

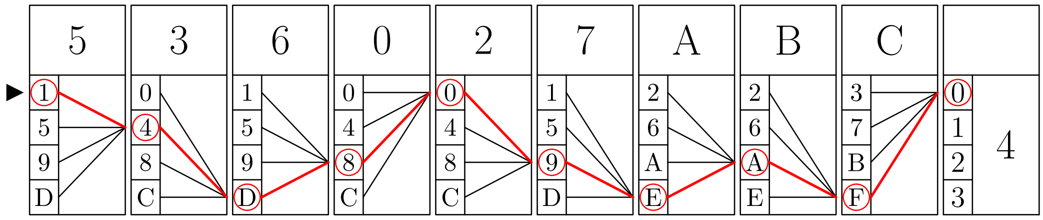

Figure 30: A division in base 16 showing only the row used.

2.3 Financial rods

In financial computations, the assumption is sometimes that the commercial year only contains 360 days, and that the daily rate is defined as the yearly rate divided by 360. This is of course an approximation, but it is a good approximation for small rates.

These assumptions make it very easy to adapt the division rods for financial computations. In fact, the financial rods are merely division rods restricted to the dividers 2, 4, 6, 8, 9 and 12 (see figure 31). The last strip, for the divider 12, is constructed like the other ones, but for clarity some lines are dashed and we have represented them as in the set originally published. Now, how are these rods used and how do they relate to rates?

If we have a yearly rate of 3%, and start with an amount of 1234 euros, then the daily rate is naturally

$$\frac{1234 \times \frac{3}{100}}{360} = \frac{\frac{1234}{12}}{1000}.$$

Thus, we see that dividing by 12 will correspond to a rate of 3%. Similarly, we would find that the dividers 9, 8, 6 and 4 correspond to rates of 4%, 4.5%, 6% and 9%. These rates were written on the index ruler. The rate corresponding to dividing by 2 is 18% and was not written for lack of space.

In the original instructions (see their reproduction in Weiss' article [39]), the example of an initial sum of 52300 francs with a rate of 4.5% is taken, and the rods yield the value 06537, which needs to be divided by 1000, resulting in a daily rate of 6 francs and 537 thousandths. These instructions also mention that the rate of 5%, although not provided by the rods, can easily be obtained by first computing the daily rate at 4%, and adding to the value found its quarter, which can be obtained by the row for 9%. For instance, if we wish to have the daily rate of 52300 francs at 5%, we would first obtain 5.811 francs (4%), and the quarter of this value is 1.452 francs (1%), which added to 5.811 yields 7.263 francs (5%). With the above definition of a daily rate, percentages can be added.

Examples of computations are shown in figures 32 and 33. In the first case, we start with the sum of 536027 euros at a rate of 9% and obtain a value of 134006, that is 134 euros and 6 thousandth for one day. In the second case, we start with 7611459040 euros at 3%, and obtain a daily rate of 634288.253 euros.

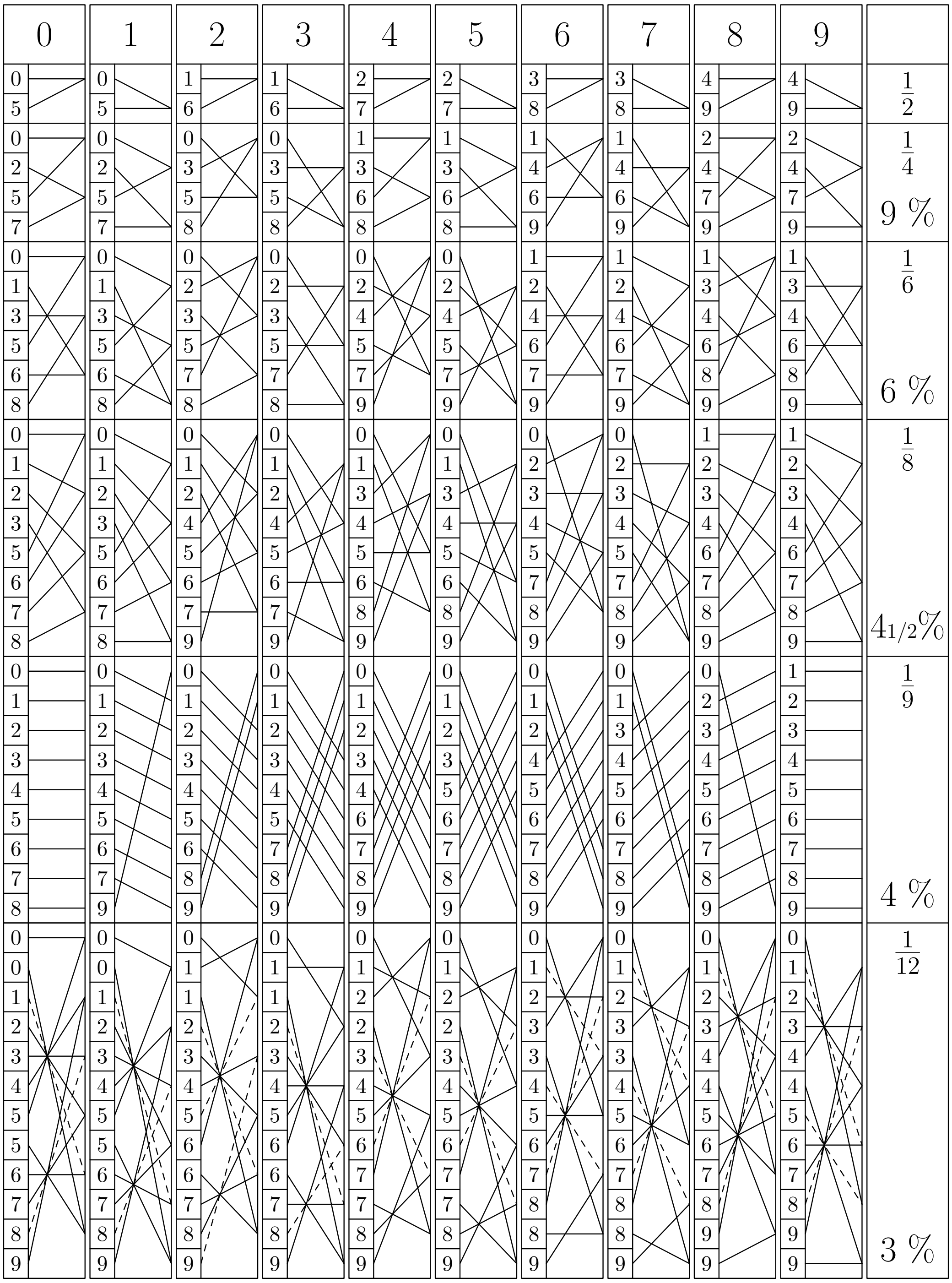

Figure 31: The financial rods.

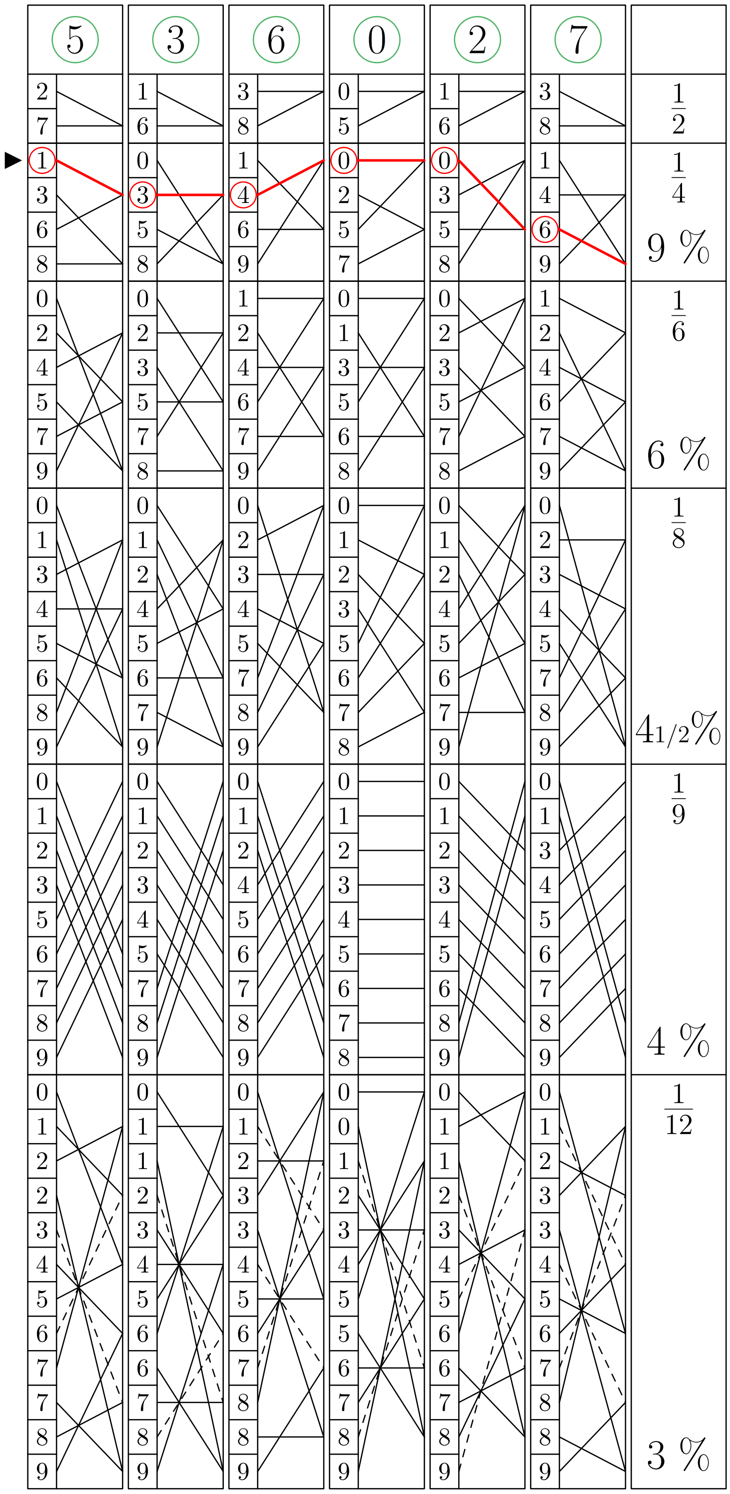

Figure 32: An example of computation.

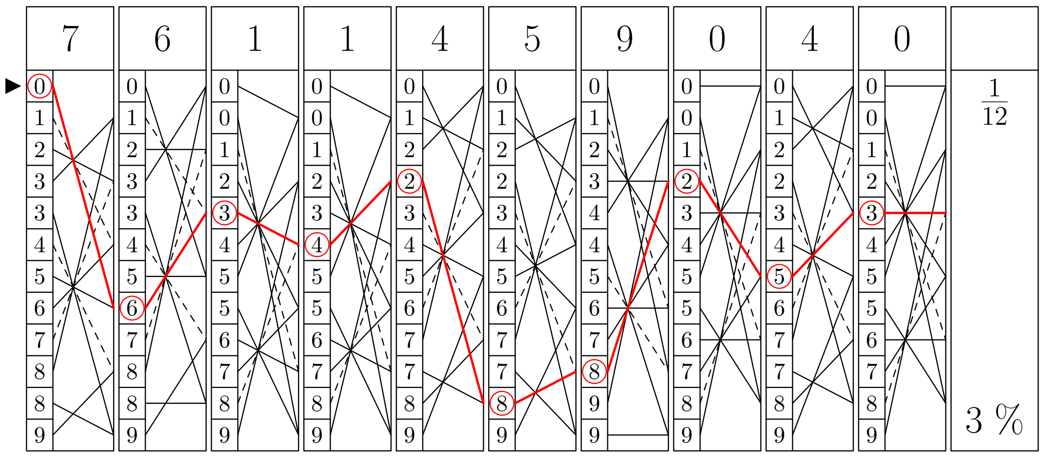

Figure 33: A computation shown in isolation.

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