

A note on the complexity of Bürigi's algorithm for the computation of sines

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1 Introduction

Folkerts, Launert and Thom have recently announced a very interesting discovery about Jost Bürigi (1552–1632) [5, 6]. Bürigi was a remarkable mechanician, clockmaker and instrument maker, and also the author of a table of progressions which can be viewed as a table of logarithms. In this note, we focus on some specific points concerning Bürigi's newly discovered algorithm.¹

2 Bürigi's algorithm

In his *Fundamentum Astronomiæ* [6], Bürigi provided an interesting and new way to compute a table of sines. Bürigi's algorithm comprises several parts, one of which allows for the simultaneous computation of the values of $\sin \frac{i\pi}{2n}$ for $0 < i < n$, and $\cos \frac{(2i+1)\pi}{4n}$ for $0 < i < n$, to any desired accuracy. However, this was only one part of Bürigi's algorithm, as Bürigi made use of other techniques to reconstruct values of sines by accumulating differences.

In this section, we briefly summarize the most original part of Bürigi's algorithm, drawing from our earlier note [10], with some improvements.

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¹For more on Bürigi's work on logarithms, we refer the reader to our critical analysis of Bürigi's work on progressions [9]. A translation of Bürigi's introduction to the tables was recently published by Kathleen Clark [3].

2.1 Dividing the quadrant in nine parts

Folkerts *et al.* [5] give the following table adapted from Bürgi's manuscript dated from the 1580s. Bürgi used instead sexagesimal values, but we can safely convert them to base 10.

	c_5		c_4		c_3		c_2		c_1
0	0		0		0		0		0
10	2,235,060	2,235,060	67,912	67,912	2,064	2,064	63	63	2
20	4,402,208	2,167,148	133,760	67,848	4,065	2,001	124	61	4
30	6,435,596	2,033,388	195,543	61,783	5,942	1,877	181	57	6
40	8,273,441	1,837,845	251,384	55,841	7,638	1,696	232	51	7
50	9,859,902	1,586,461	299,587	48,203	9,102	1,464	276	44	8
60	11,146,776	1,286,874	338,688	39,101	10,290	1,188	312	36	9
70	12,094,962	948,186	367,499	28,811	11,166	876	339	27	10
80	12,675,649	580,687	385,144	17,645	11,703	537	356	17	11
90	12,871,192	195,543	391,086	5,942	11,884	181	362	6	12

The purpose of this table is to compute the values of $\sin 10^\circ$, $\sin 20^\circ$, \dots , $\sin 90^\circ$, to any desired accuracy. Bürgi's algorithm is deceptively simple. There is no bisection, no roots, only two basic operations: many additions, and a few divisions by 2. The computations can be done with integers, or with sexagesimal values, or in any other base.

In order to compute the sines, Bürgi starts with an arbitrary list of values, which can be considered as first approximations of the sines, but need not be. These values are given in column c_1 . In all these columns, c_1 to c_5 , the last value is always the *sinus totus*. That is, the first approximation starts with $\sin 90^\circ = 12$. This gives in modern terms for $\sin 60^\circ$ the value $\frac{9}{12} = 0.75$, an approximation of the actual sine which is $0.866\dots$. It basically matters very little with what values one starts. It is even possible to take all initial values equal to 1, for instance, or in decreasing order, or at random, but they can't all be taken equal to 0, except if the last one is equal to 2 (when working with integers). If real values are used, it would work even if all values are equal to 0, and the last one is equal to 1. Negative values do also work, but some distributions will fail, for instance with values that cancel each other, such as -1 , followed by 1.

Column c_5 shows the result of that algorithm, here conducted to four steps, but the table could have been extended at will towards the left. Now, in column c_5 , there is a new value for the *sinus totus*, namely 12871192, and therefore we have as a new approximation of $\sin 60^\circ$ the fraction $\frac{11146776}{12871192} = 0.86602515136$, the exact value being $\frac{\sqrt{3}}{2} = 0.866025403\dots$

Bürge's algorithm is an iterative procedure for computing all the values of column c_{i+1} from those of column c_i . The computations use an intermediate column whose last value is half of the previous *sinus totus*. This intermediate column actually provides the values of the cosines, but for intermediate angles, here 5° , 15° , \dots , 85° . In the above example, the *sinus totus* in column c_1 is 12, and the last value of the column between c_2 and c_1 is 6. The last value of the column between c_3 and c_2 is 181, half of 362. If the *sinus totus* is odd, one might take the exact half, but it actually does not matter, as this algorithm leads to increasingly larger numbers, and ignoring a half integer only has marginal consequences on the convergence.

Once the last value of an intermediate column has been obtained, all other values of that column are obtained by adding the values in the previous column, as if the previous column were differences. So, we have $6 + 11 = 17$, $17 + 10 = 27$, and so on.

When the intermediate column has been filled, the new column c_{i+1} is constructed by starting with 0, and adding the values of the intermediate column.

And that's all!

2.2 Bürge's path to his algorithm

Bürge did not explain how he obtained his algorithm, but in a recent article, we suggested a very simple way [10]. Recently, other constructions have been provided. In particular, one was announced by Peter Ullrich and will be disclosed in March 2016, and a geometric version of Thom's proof was recently given by Christian Riedweg [7].

3 The accuracy of Bürge's algorithm

Bürge seems to have computed a sine table for every $2''$ with 8 places. Whether these 8 places were decimal or sexagesimal, we do not know. We also do not know how accurate this table was, since it is no longer extant. A number of large tables of logarithms have a number of erroneous figures, so that it would be a hastily drawn conclusion to state that Bürge really had computed a table of sines accurate to 8 places.

Moreover, it seems accepted that Bürge only used the above algorithm for a number of pivot values, so that eventually smaller gaps could be bridged by adding differences.

It is however interesting to examine the behavior of Bürge's main algorithm and to somehow measure how it does improve the accuracy of sines.

In their article, Folkerts, Launert and Thom stated that for the division of the quadrant in nine parts, the sines were correct to 6 or 7 places, and that an accuracy of 8 (decimal) places can be obtained by adding three more columns [5]. The more general question, which was not answered by Folkerts, Launert and Thom, is how far one has to go to achieve a certain accuracy.

In order to obtain a partial answer to this question, we considered a special case. First, we assume that the quadrant is divided in n parts, so that the purpose is to compute the values of $\sin \frac{j\pi}{2n}$ for $0 \leq j \leq n$. In the above, we have named the columns c_1, c_2 , etc., but now we also name the intermediate (auxiliary) columns a_1, a_2 , etc. Within c_i , the values from top to bottom are $c_{i,0}, c_{i,1}, c_{i,2}, \dots, c_{i,n}$, and within a_i , the values are $a_{i,0}, a_{i,1}, a_{i,2}, \dots, a_{i,n-1}$. We have therefore

$$a_{i,n-1} = bc_{i,n}/2c \quad (1)$$

$$a_{i,j} = a_{i,j+1} + c_{i,j+1} \text{ for } 0 \leq j < n-1 \quad (2)$$

$$c_{i,0} = 0 \quad (3)$$

$$c_{i,j} = c_{i,j-1} + a_{i-1,j-2} \text{ for } 0 < j \leq n \quad (4)$$

For a given i , $c_{i,j}/c_{i,n}$ is an approximation of $\sin \frac{j\pi}{2n}$. But how good is this approximation? In order to answer this question, we have made some experiments, first with the initial values:

$$c_{0,j} = j \text{ for } 0 \leq j \leq n \quad (5)$$

that is, with a linear approximation of the sines, but then also with a number of other distributions, including

$$c_{0,j} = 1 \text{ for } 0 \leq j \leq n, \quad (6)$$

$$c_{0,j} = 0 \text{ for } 0 \leq j < n \text{ and } c_{0,n} = 2 \quad (7)$$

and

$$c_{0,j} = n - j \text{ for } 0 \leq j \leq n \quad (8)$$

Now, given such a distribution and the above algorithm, we can define the error on the j -th value for a given i :

$$e_n(i, j) = \sin \frac{j\pi}{2n} - \frac{c_{i,j}}{c_{i,n}} \quad (9)$$

as well as the maximal error in a column:

$$E_n(i) = \max_{j \in \{0, \dots, n\}} f_{e_n}(i, j) \quad (10)$$

Moreover, we can define the position of the greatest error:

$$G_n(i) = \min_{j \in \{0, \dots, n\}} f_{e_n}(i, j) = E_n(i) \quad (11)$$

We have only taken the minimum value in order to ensure that there is only one integer value.

These two definitions lead us to the following conjectures:

Conjecture 1 *The greatest error occurs close to $\arcsin(1/\sqrt{3})$, so that we conjecture that*

$$\lim_{i \rightarrow \infty} \frac{G_n(i)}{n} = \frac{2}{\pi} \arcsin \frac{1}{\sqrt{3}} + \nu(n)$$

where $\lim_{n \rightarrow \infty} \nu(n) = 0$.

Conjecture 2

$$\lim_{i \rightarrow \infty} \frac{E_n(i+1)}{E_n(i)} = \frac{1}{9} + \xi(n)$$

where $\lim_{n \rightarrow \infty} \xi(n) = 0$.

Assuming that these conjectures are true,² we can now estimate the number of steps needed to achieve a certain accuracy p , namely $p/\log 9 \approx 1.05p$. Without too much error, we can even assume that the maximal error is divided by 10 for every step, so that each column practically gives a new place of accuracy, or nearly so.

4 The complexity of Bürgi's algorithm

Another question which was not tackled by Folkerts, Lauenert and Thom is that of the complexity of Bürgi's main algorithm. In other words, how much effort does it take to reach a certain accuracy? From the previous section, we now know that each new step gives a little less than a new place of accuracy. But does this represent a lot of work, or not? In their article, Folkerts, Lauenert and Thom state that continuing the computations for $n = 9$ up to column 8 is still "a reasonable computational effort." But what if we do divide the right angle in 90 parts? Or 5400 parts? Or more?

²(Note added 27 march 2016) With only minor exceptions, these conjectures appear to follow from the analysis of Bürgi's iteration as a *power iteration* scheme, as demonstrated recently by Jörg Waldvogel [11].

Even though Burigi did not ambition to use his main algorithm to compute his entire sine table, it is theoretically possible. Can we estimate how much time it would have taken? Let us try.

Our aim is to obtain a gross estimate of the complexity of filling the table of section 2.1. The rightmost column c_1 is assumed to be filled, but a_1, c_2, a_2, c_3 , etc., need to be computed. We can estimate the number of one-digit additions as being approximately the number of digits in all these columns. Consider for instance column c_5 . The values are approximately $k \sin \frac{j\pi}{18}$ with $k = \mu^4 = 12$, where $\mu = \frac{\csc^2(\pi/(4n))}{4}$. In general, in our setting, $k = \mu^4 = n$, because $c_{1,n} = n$. $a_{i,j}$ can also be approximated using the fact that the first differences of the sines are close to the cosines and that $c_{i+1,1} = a_{i,0}$. We obtain:

$$c_{i,j} = \mu^{i-1} n \sin \frac{j\pi}{2n} \quad (12)$$

$$a_{i,j} = \mu^i n \cos \frac{j\pi}{2n} = \sin \frac{\pi}{2n} \quad (13)$$

We stress that these are gross approximations, but they are sufficient for our purpose.

Now, the number of digits of $c_{i,j}$ is approximated by $\log(c_{i,j})$ and since we want to count all these digits, we need to compute

$$d_c(i) = \sum_{j=0}^{j \neq n} \log(c_{i,j}) \quad (14)$$

which is approximated by an integral:

$$d_c(i) = \frac{\int_0^{\pi/2} \log(k \sin x) dx}{\frac{\pi}{2n}} \quad (15)$$

Using the fact that

$$\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2 \quad (16)$$

we arrive at

$$d_c(i) = n \log \frac{\mu^{i-1} n}{2} \quad (17)$$

Taking for instance $i = 5$, $n = 9$ and $\mu = 32.9\dots$, we find $d_c(5) = 60$, which is close to the actual number of digits in column c_5 .

Using a similar reasoning, we obtain the number of digits d_a of the intermediate columns:

$$d_a(i) = n \log \frac{\mu^i - n \sin(\pi/2n)}{2} \quad (18)$$

Adding all these terms and keeping only the largest factors, we obtain an approximation for the total number of digits $D(p, n)$ in columns $a_1, c_2, a_3, \dots, c_p$:

$$D(p, n) = 2p^2 n \log n \quad (19)$$

where $\log n$ is the base 10 logarithm.

For instance, in the above example, with $p = 5$ and $n = 9$, we obtain $D(5, 9) = 429$. The actual value is about 340, but, as we said, $D(p, n)$ is only a gross approximation.

Now, if we take for instance $n = 90$ and $p = 8$, we obtain about 22500 digits, and about that many one-digit additions. With $n = 5400$ and $p = 8$, we obtain about 2.6 million digits. And for a 2" sine table to 8 places of accuracy, we have about 100 million digits. Assuming that about 10 one-digit additions can be done per minute, even $n = 5400$ seems out of reach for one person, and $n = 90$ may take perhaps 40 hours to one calculator. This shows, in our opinion, that although Bürgi's algorithm is very simple, it has to be adapted or used in conjunction with other methods. Bürgi had certainly found out about the limitations of his main algorithm, and his genius in fact lies in the clever combination of several algorithms.

5 Accelerating the convergence

The main problem with Bürgi's algorithm is that the values of the table quickly become very large. On the other hand, if one's purpose is only to obtain the sines to a certain accuracy, it is in fact not necessary to manipulate the actual large numbers. This was noticed, we believe, by the author of the manuscript page reproduced by Launert [6, p. 57], who in the last steps of the computation, truncated the large values he was manipulating.³

³Incidentally, Launert suggests that the author of this page inserted in the Leiden copy of Ursus's *Fundamentum Astronomicum* may be John Bainbridge, and tries therefore (with the help of an annotation that vaguely looks like "H. Briggs") to establish a connection, or an independent discovery of Bürgi's algorithm by Briggs. This connection is also made in

Therefore, if we intend to find the sines to 8 places, we could merely truncate all the values of the *sinus totus* to at most 9 figures. In that case, however, the conjectures given above are no longer valid, since they assume that the computations are done on integers and with no truncation.

Here too, we can obtain a gross estimate of the computational complexity of the calculation. Assuming that we want to obtain the sines with p figures, we simplify the calculations by using numbers in the same ranges in all columns. That is, we start for instance with a linear distribution

$$c_{1,i} = \frac{i}{n} 10^{p+1} \quad (20)$$

and we compute p new pairs of columns. This will give the sines to p places with a good accuracy.⁴

Now, if we have computed the column c_i , the last value of column c_{i+1} will approximately be $\mu 10^{p+1}$ (it could be up to ten times more, but this is an estimate). The first value of column a_i will be about $\frac{\rho}{\mu} 10^{p+1}$. Now, from these maximal values, we can derive estimates for the number of digits, merely adapting the expressions given above:

$$d_c(i) = d_c + n \log \frac{\mu 10^{p+1}}{2} \quad (21)$$

$$d_a(i) = d_a + n \log \frac{\frac{\rho}{\mu} 10^{p+1}}{2} \quad (22)$$

and the overall complexity will grossly be

the second version, dated February 2, 2016, of Folkerts, Launert and Thom's article [5]. We believe that such an interpretation is very far-fetched, and that it is unlikely that two persons would come up independently with such an algorithm. Instead, we believe that the annotation which may look like "H. Briggs" actually directly refers to Burgi, and that the author of the page wanted to stress clearly that Burgi is the author of the algorithm. Of course, we do not know who really wrote this page, and how he came to know about this algorithm, but it seems to us the most plausible explanation. Launert also seems to allude to a possible reuse of Burgi's methods by Briggs when the latter computed the sines later used in his *Trigonometria Britannica* (1633) [1]. We do not exclude the possibility that Briggs knew of Burgi's algorithm, but since he does not mention it, nor uses any iterative algorithm for the computation of a set of sines, such a reuse seems doubtful.

⁴One should be aware, however, that the truncations will introduce fluctuations in the accuracy, and that the values of the sines can sometimes become more accurate from one column to the next, and sometimes less accurate. Depending on how the truncation is performed, and where exactly one stops, the results will vary. The ratios obtained are in general not the best ratios approximating the set of sines, but merely a set of ratios with a bounded error.

$$D(p, n) \approx p \cdot (d_c + d_a) \tag{23}$$

$$pn \log \frac{\mu^{\frac{p}{\mu}}}{\mu} \cdot \frac{10^{2p+2}}{4} \tag{24}$$

$$3pn \log n \tag{25}$$

This is now somewhat more interesting than the exact computation with large integers. Taking again the previous examples, for $n = 90$ and $p = 8$, we obtain about 4000 digits. For $n = 5400$ and $p = 8$, we obtain about 500000 digits, and for a 2'' sine table to 8 places of accuracy, we have about 20 million digits. $n = 90$ may take perhaps 8 hours to one calculator, and $n = 5400$ may take about 800 hours. It is still basically untractable, but less so than in the initial version of the algorithm. We stress again that these are all estimates based on various approximations, but they should give a general idea of the improvement in the performance of the algorithm.

6 Conclusion

In this note, we have examined a number of important aspects of Bürgi's algorithm for the computation of sines. We have shed some light on how fast a certain accuracy can be obtained and the necessity to adapt Bürgi's algorithm and to use it in conjunction with other methods. But whether Bürgi did in fact compute a table to 8 (decimal) places of accuracy is still not known...

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